Review exercise 1

1 The inequality can be solved as follows:

$$\frac{2}{x-2}<\frac{1}{x+1}$$

$$\frac{2}{x-2}-\frac{1}{x+1}<0$$

$$\frac{2x+2-x+2}{(x-2)(x+1)}<0$$

$$\frac{x+4}{(x-2)(x+1)}<0$$

The inequality is satisfied when numerator and denominator do not have the same sign. The numerator is positive for $x>-4$, while the denominator is positive for $x>2$ or $x<-1$

Therefore, the inequality holds for $x<-4$ or for $-1<x<2$

2

$$\frac{x^2}{x-2}>2x$$

$$\frac{x^2}{x-2}-2x>0$$

$$\frac{x^2-2x(x-2)}{x-2}>0$$

$$\frac{4x-x^2}{x-2}>0$$

$$\frac{x(4-x)}{x-2}>0$$

You collect the terms together on one side of the inequality, write the expression as a single fraction and factorise the result as far as possible.

Considering $f(x)=\dfrac{x(4-x)}{x-2}$,

the critical values are x = 0, 2 and 4

You find the critical values by solving the numerator equal to zero and the denominator equal to zero. In this case the numerator = 0, gives x = 0, 4 and the denominator gives x = 2

	$x<0$	$0<x<2$	$2<x<4$	$4<x$
Sign of f(x)	+	–	+	–

For example if $4<x$, then $\dfrac{x(4-x)}{x-2}=\dfrac{\text{positive}\times\text{negative}}{\text{positive}}$, which is negative.

The solution of $\dfrac{x^2}{x-2}>2x$ is

$\{x: x<0\}\cup\{x: 2<x<4\}$

3 $\dfrac{x^2-12}{x} > 1$

Multiply both sides by x^2

x cannot be zero as $\dfrac{x^2-12}{x}$ would be undefined, so x^2 is positive and you can multiply both sides of an inequality by a positive number or expression without changing the inequality. You could **not** multiply both sides of the inequality by x as x could be positive or negative.

$$\frac{x^2-12}{\cancel{x}} \times x^{\cancel{2}} > x^2$$

$$x(x^2-12) - x^2 > 0$$

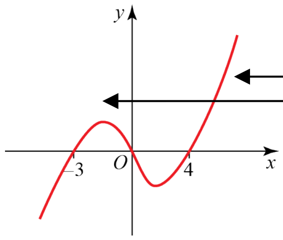
$$x^3 - 12x - x^2 > 0$$

$$x(x^2 - x - 12) > 0$$

$$x(x-4)(x+3) > 0$$

Sketching $y = x(x-4)(x+3)$

The graph of $y = x(x-4)(x+3)$ crosses the y axis at $x = -3$, 0 and 4

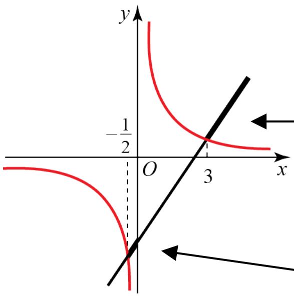


You can see from the sketch that the graph is above the x-axis for $-3 < x < 0$ and $x > 4$
You can then just write down this answer.

The solution of $\dfrac{x^2-12}{x} > 1$ is $\{x : -3 < x < 0\} \cup \{x : x > 4\}$

If you preferred, you could solve this question using the method illustrated in the solutions to questions 2 and 3 above.

4



Both $y = 2x - 5$ and $y = \dfrac{3}{x}$ are straightforward graphs to sketch and so this is a suitable question for a graphical method. The question, however, specifies no method and so you can use any method which gives an exact

After sketching the two graphs, $2x - 5 > \dfrac{3}{x}$ is the set of values of x for which the line is above the curve. These parts of the line have been drawn thickly on the sketch.

$$2x - 5 = \frac{3}{x}$$

$$x(2x-5) = 3$$

$$2x^2 - 5x - 3 = 0$$

$$(2x+1)(x-3) = 0$$

$$x = -\tfrac{1}{2}, 3$$

You need to find the x-coordinates of the points where the line and curve meet to find two end points of the intervals. The other end point ($x = 0$) can be seen by inspecting the sketch.

The solution to $2x - 5 > \dfrac{3}{x}$ is $\{x : -\tfrac{1}{2} < x < 0\} \cup \{x : x > 3\}$

5 $$\frac{x+k}{x+4k} > \frac{k}{x}$$

$$\frac{x+k}{x+4k} - \frac{k}{x} > 0$$

$$\frac{(x+k)x - k(x+4k)}{(x+4k)x} > 0$$

$$\frac{x^2 - 4k^2}{(x+4k)x} > 0$$

$$\frac{(x+2k)(x-2k)}{(x+4k)x} > 0$$

Considering $\mathrm{f}(x) = \dfrac{(x+2k)(x-2k)}{(x+4k)x}$,

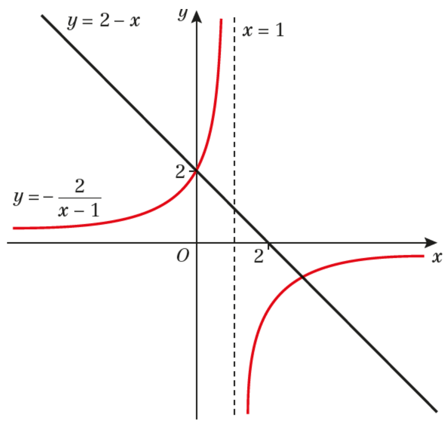
the critical values are $x = -4k, -2k, 0$ and $2k$

For example, when k is positive, in the interval $0 < x < 2k$, $\dfrac{(x+2k)(x-2k)}{(x+4k)x} = \dfrac{\text{positive} \times \text{negative}}{\text{positive} \times \text{positive}}$, which is

	$x < -4k$	$-4k < x < -2k$	$-2k < x < 0$	$0 < x < 2k$	$2k < x$
Sign of f(x)	+	–	+	–	+

The solution of $\dfrac{x+k}{x+4k} > \dfrac{k}{x}$ is $\{x : x < -4k\} \cup \{x : -2k < x < 0\} \cup \{x : x < 0\} > 2k$

6 a



b The points of intersection are those whose x-coordinate satisfies the equation $2 - x = -\dfrac{2}{x-1}$

We solve this:

$(2-x)(x-1) = -2$

$(x-2)(x-1) = 2$

$x^2 - 3x + 2 = 2$

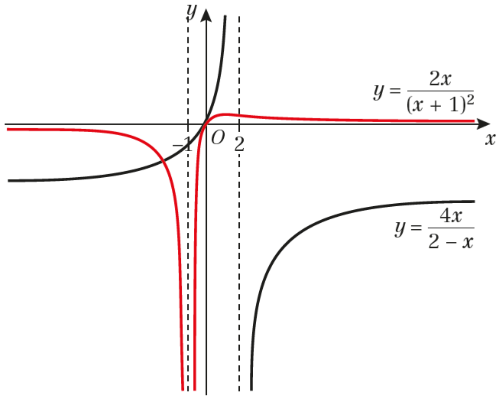
$x^2 - 3x = 0$

which is solved by $x = 0$ and $x = 3$

Both solutions are acceptable. Therefore, the points of intersection are (0, 2) and $(3, -1)$

c It is clear from the graph that the solution to the inequality is $x < 0$ or $1 < x < 3$

7 a



b The points of intersection are those whose x-coordinate satisfies the equation $\dfrac{4x}{2-x} = \dfrac{2x}{(x+1)^2}$

We solve this:

$4x(x+1)^2 = 2x(2-x)$

$4x(x^2+2x+1) = 4x - 2x^2$

$4x^3 + 8x^2 + 4x = 4x - 2x^2$

$4x^3 + 10x^2 = 0$

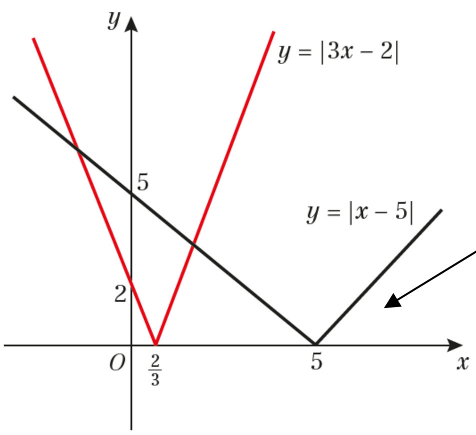
$x^2(2x+5) = 0$

Which is solved by $x = 0$ and $x = -\frac{5}{2}$

These are both acceptable solutions. Therefore, the points of intersection are $(0,0)$ and $\left(-\frac{5}{2}, -\frac{20}{9}\right)$

c It is clear from the graph that the set of the solutions to the inequality is $\left\{x : x \leqslant -\frac{5}{2}\right\} \cup \left\{x : x = 0\right\} \cup \left\{x : x > 2\right\}$

8 a



You should mark the coordinates of the points where the graphs meet the axes.

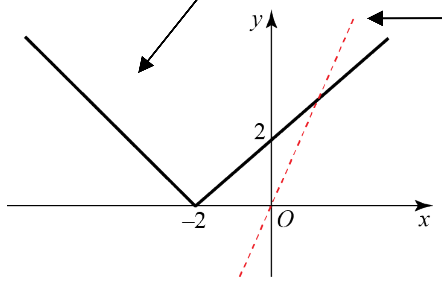
Inequalities which contain both an expression in x with a modulus sign and an expression in x without a modulus sign, are usually best answered by drawing a sketch. In this case, you have been instructed to draw the sketch first. The continuous line is the graph of $y = |x+2|$

You should mark the coordinates of the points where the graph cuts the axis.

b The points of intersection are those whose x-coordinate satisfies the equation $|x-5|=|3x-2|$
For $x>5$ or $x<\frac{2}{3}$, this is equivalent to $x-5=3x-2$, which is solved by $x=-\frac{3}{2}$, which is acceptable. For $\frac{2}{3}<x<5$, it is equivalent to $5-x=3x-2$, which is solved by $x=\frac{7}{4}$, which is also acceptable. Therefore, there are two points of intersection: $\left(-\frac{3}{2},\frac{13}{2}\right)$ and $\left(\frac{7}{4},\frac{13}{4}\right)$

8 c It is clear from the graph that the set of the solutions to the inequality is $\left\{x:x<-\frac{3}{2}\right\}\cup\left\{x:x>\frac{7}{4}\right\}$

9 a



Inequalities which contain both an expression in x with a modulus sign and an expression in x without a modulus sign, are usually best answered by drawing a sketch. In this case, you have been instructed to draw the sketch first. The continuous line is the graph of $y=|x+2|$

You should mark the coordinates of the points where the graph cuts the axis.

You should now add the graph $y=2x$ to your sketch. This has been done with a dotted line. You find the solution to the inequality by identifying the values of x where the dotted line is above the continuous line.

b The intersection occurs when $x>-2$

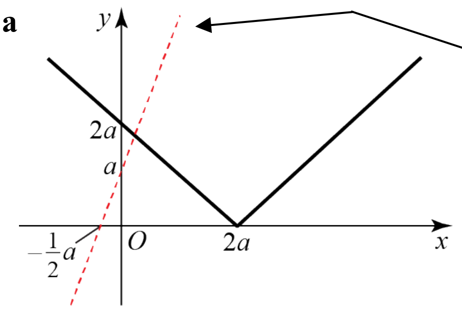
When $-2, |x+2|=x+2$

$$2x=x+2$$
$$x=2$$

The solution of $2x>|x+2|$ is $x>2$

When f(x) is positive, $|\mathrm{f}(x)|=\mathrm{f}(x)$

10 a



The dotted line is added to the sketch in part **a** to help you to solve part **b**. The dotted line is the graph of $y = 2x + a$ and the solution to the Inequality in part **b** is found by identifying where the continuous line, which corresponds to $|x-2a|$, is above the dotted line, which corresponds to $2x + a$

b The intersection occurs when $x < 2a$

When $x<2a, |x-2a|=2a-x$

$$2a-x=2x+a$$
$$-3x=-a\Rightarrow x=\tfrac{1}{3}a$$

The solution of $|x-2a|>2x+a$ is $x<\frac{1}{3}a$

If f(x) is negative, then $|\mathrm{f}(x)|=-\mathrm{f}(x)$

11 We have two cases, depending on the sign of $\dfrac{x}{x-3}$

If $x>3$ or $x<0$, then it is positive and the inequality becomes $\dfrac{x}{x-3}<8-x$, which can be solved as follows:

$$\frac{x}{x-3}+x-8<0$$

$$\frac{x+x^2-3x-8x+24}{x-3}<0$$

$$\frac{x^2-10x+24}{x-3}<0$$

$$\frac{(x-6)(x-4)}{x-3}<0$$

This holds when numerator and denominator do not have the same sign; the numerator is positive for $x>6$ or $x<4$, while the denominator is positive for $x>3$

Therefore, this inequality is solved for $x<3$ or for $4<x<6$; we were looking for solutions with $x>3$ or $x<0$, therefore the set satisfying the first case is $\{x: x<0\}\cup\{x: 4<x<6\}$

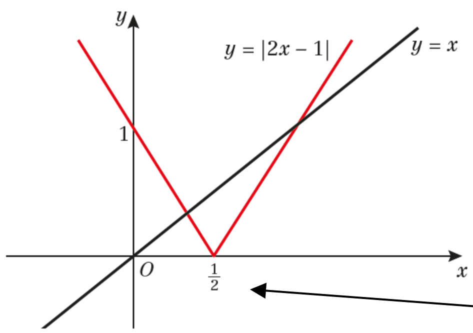
In the second case, we have $0<x<3$

Here, the inequality becomes $\dfrac{x}{3-x}<8-x$, which leads to $\dfrac{x^2-12x+24}{x-3}<0$

This is solved by $x<6-2\sqrt{3}$ or $x>3$

Clearly only the first one of these solutions is acceptable. Therefore, the set of the solutions is $\{x: 4<x<6\}\cup\{x: x<6-2\sqrt{3}\}$

12 a



At $x=0, y=|-1|=1$

At $x=\frac{1}{2}, y=0$

b There are two points of intersection.

At the right hand point of intersection,

$x>\frac{1}{2}\Rightarrow|2x-1|=2x-1$

$2x-1=x\Rightarrow x=1$

At the left hand point of intersection,

$x<\frac{1}{2}\Rightarrow|2x-1|=1-2x$

$1-2x=x\Rightarrow x=\frac{1}{3}$

If $\mathrm{f}(x)>0$, then $|\mathrm{f}(x)|=\mathrm{f}(x)$

If $\mathrm{f}(x)<0$, then $|\mathrm{f}(x)|=-\mathrm{f}(x)$

The points of intersection of the two graphs are

$\left(\frac{1}{3},\frac{1}{3}\right)$ and $(1, 1)$

You need to give both the x-coordinates and the y-coordinates.

c The solution of $|2x-1|>x$ is $\{x: x<\frac{1}{3}\}\cup\{x: x>1\}$

13
$$|x-3|>2|x+1|$$
$$(x-3)^2>4(x+1)^2$$
$$x^2-6x+9>4x^2+8x+4$$
$$0>3x^2+14x-5$$
$$(x+5)(3x-1)<0$$

As both $|x-3|$ and $2|x+1|$ are positive you can square both sides of the inequality without changing the direction of the inequality sign. If a and b are both positive, it is true that
$a>b \Rightarrow a^2>b^2$
You cannot make this step if either or both of a and b are negative.

Considering $f(x)=(x+5)(3x-1)$, the critical values are $x=-5$ and $\frac{1}{3}$

Alternatively you can draw a sketch of $y=(x+5)(3x-1)$ and identify the region where the curve is below the y-axis.

	$x<-5$	$-5<x<\frac{1}{3}$	$\frac{1}{3}<x$
Sign of f(x)	+	–	+

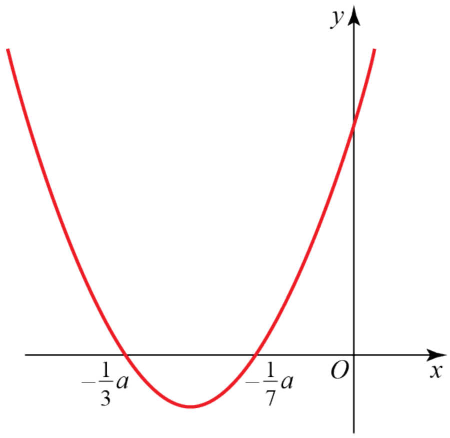
The solution of $|x-3|>2|x+1|$ is
$\{x: -5<x<\frac{1}{3}\}$

14
$$|5x+a|\leqslant|2x|$$
$$(5x+a)^2\leqslant(2x)^2$$
$$25x^2+10ax+a^2\leqslant4x^2$$
$$21x^2+10ax+a^2\leqslant0$$
$$(3x+a)(7x+a)\leqslant0$$

As a is positive, both $|5x+a|$ and $|2x|$ are positive and you can square both sides of the inequality.

Sketching $y=(3x+a)(7x+a)$

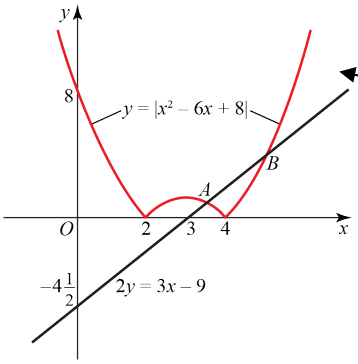
The graph is a parabola intersecting the x-axis at $x=-\frac{1}{3}a$ and $x=-\frac{1}{7}a$



A common error here is not to realise that, for a positive $a, -\frac{1}{3}a$ is a smaller number than $-\frac{1}{7}a$
It is very easy to get the inequality the wrong way round.

The solution of $|5x+a|\leqslant|2x|$ is $-\frac{1}{3}a\leqslant x\leqslant-\frac{1}{7}a$

15 a



As $x^2-6x+8=(x-2)(x-4)$ the curve meets the x-axis at $x = 2$ and $x = 4$
The sketching of the graphs of modulus functions is in Chapter 5 of book C3

The curve meets the x-axis at (2, 0) and (4, 0)
The line meets the x-axis at (3, 0)

15 b To find the coordinates of A. The x-coordinate of A is in the interval $2 < x < 4$

In this interval $x^2 – 6x + 8$ is negative and, hence,

$$\left|x^2-6x+8\right| = -x^2+6x-8$$

If f(x) < 0, then $\left|f(x)\right| = -f(x)$

$$-x^2+6x-8=\frac{3x-9}{2}$$
$$-2x^2+12x-16=3x-9$$
$$2x^2-9x+7=0$$
$$(2x-7)(x-1)=0$$
$$x=\tfrac{7}{2},\cancel{1}$$
$$y=\frac{3\times\frac{7}{2}-9}{2}=\frac{3}{4}$$

As the x-coordinate of A is in the interval $2 < x < 4$, the solutions $x = 1$ must be rejected.

The coordinates of A are $\left(\tfrac{7}{2},\tfrac{3}{4}\right)$,

To find the coordinates of B. The x-coordinate of B is in the interval $x > 4$

In this interval x^2-6x+8 is positive and, hence,

$$\left|x^2-6x+8\right| = x^2-6x-8$$

If f(x) > 0, then $\left|f(x)\right| =$ f(x)

$$x^2+6x+8=\frac{3x-9}{2}$$
$$2x^2-12x+16=3x-9$$
$$2x^2-15x+25=0$$
$$(x-5)(2x-5)=0$$
$$x=5,\cancel{\tfrac{5}{2}}$$
$$y=\frac{3\times5-9}{2}=3$$

As the x-coordinate of B is in the interval $x > 4$, the solution $x=2\frac{1}{2}$ must be rejected.

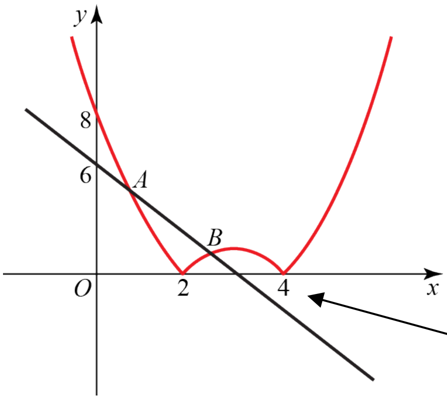
The coordinates of B are (5, 3)

The solution of $2\left|x^2-6x+8\right|>3x-9$ is

$$x<\tfrac{7}{2}, x>5$$

You solve the inequality by inspecting the graphs. You look for the values of x where the curve is above the line.

16 a



You should mark the coordinates of the points where the graphs meet the axes.

16 b Let the points where the graphs intersect be A and B

For A, $(x-2)(x-4)$ is positive

$$(x-2)(x-4)=6-2x$$
$$x^2-6x+8=6-2x$$
$$x^2-4x=-2$$
$$x^2-4x+4=2$$
$$(x-2)^2=2$$
$$x=2-\sqrt{2}$$

The quadratic equations have been solved by completing the square. You could use the formula for solving a quadratic but the conditions of the question require exact solutions and you should not use decimals.

The quadratic equation has another solution $2+\sqrt{2}$ but the diagram shows that the x-coordinate of A is less than 2, so this solution is rejected.

For B, $(x-2)(x-4)$ is negative

$$-(x-2)(x-4)=6-2x$$
$$-x^2+6x-8=6-2x$$
$$x^2-8x=-14$$
$$x^2-8x+16=2$$
$$(x-4)^2=2$$
$$x=4-\sqrt{2}$$

The quadratic equation has another solution $2+\sqrt{2}$ but the diagram shows that the x-coordinate of B is less than 4, so this solution is rejected.

The values of x for which $|(x-2)(x-4)|=6-2x$ are $2-\sqrt{2}$ and $4-\sqrt{2}$

c The solution of $|(x-2)(x-4)|<6-2x$ is $2-\sqrt{2}<x<4-\sqrt{2}$

You look for the values of x where the curve is below the line.

17 a For $x > -2$, $x + 2$ is positive and the equation is

$$\frac{x^2-1}{x+2} = 3(1-x)$$

$$x^2 - 1 = 3(1-x)(x+2) = -3x^2 - 3x + 6$$

$$4x^2 + 3x - 7 = (4x+7)(x-1) = 0$$

$$x = -\tfrac{7}{4}, 1$$

As both of these answers are greater than –2 both are valid.

For $x < -2$, $x + 2$ is negative and the equation is

$$\frac{x^2-1}{-(x+2)} = 3(1-x)$$

$$x^2 - 1 = -3(1-x)(x+2) = 3x^2 + 3x - 6$$

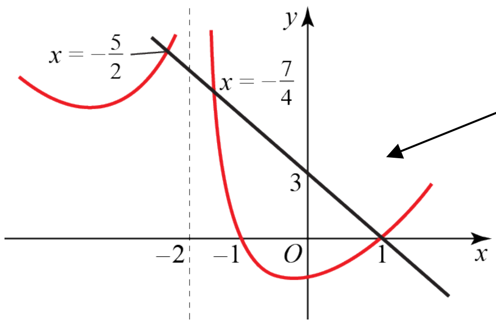
$$2x^2 + 3x - 5 = (2x+5)(x-1) = 0$$

$$x = -\tfrac{5}{2}, \cancel{1}$$

As 1 is not less than –2 the answer 1 should be 'rejected' here. However, the earlier working has already shown 1 to be a correct solution.

The solutions are $-\frac{5}{2}, -\frac{7}{4}$ and 1

17 b



To complete the question, you add the graph of y = 3(1 – x) to the graph which has already been drawn for you. You know the x-coordinates of the points of intersection from part **a**.

The solution of $\frac{x^2-1}{|x+2|} < 3(1-x)$ is

$\{x : x < -\frac{5}{2}\} \cup \{x : -\frac{7}{4} < x < 1\}$

You look for the values of x on the graph where the curve is below the line.

18 $\dfrac{2}{(r+1)(r+2)} = \dfrac{A}{r+1} + \dfrac{B}{r+2}$

$2 = A(r+2) + B(r+1)$

$2 = A = -B$

Let $f(r) = \dfrac{1}{r+1}$

$$\sum_{r=1}^{n} \frac{2}{(r+1)(r+2)} = 2\sum_{r=1}^{n}\left(\frac{1}{r+1} - \frac{1}{r+2}\right)$$

$$= 2(f(1) - f(n+1))$$

$$= 2\left(\frac{1}{2} - \frac{1}{n+2}\right)$$

$$= 1 - \frac{2}{n+2} = \frac{n}{n+2}$$

19 $\dfrac{2}{(r+1)(r+3)} = \dfrac{A}{r+1} + \dfrac{B}{r+3}$

$2 = A(r+3) + B(r+1)$

$2 = 2A = -2B$

Let $\mathrm{f}(r) = \dfrac{1}{r+1}$

$$\begin{aligned}\sum_{r=1}^{n} \frac{2}{(r+1)(r+3)} &= \sum_{r=1}^{n} \left(\frac{1}{r+1} - \frac{1}{r+3}\right) \\ &= \mathrm{f}(1) + \mathrm{f}(2) - \mathrm{f}(n+1) - \mathrm{f}(n+2) \\ &= \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3} \\ &= \frac{5(n+2)(n+3) - 6(n+3) - 6(n+2)}{6(n+2)(n+3)} \\ &= \frac{n(5n+13)}{6(n+2)(n+3)}\end{aligned}$$

Hence $a = 5, b = 13$ and $c = 6$

20 a
$$\begin{aligned}\text{LHS} &= \frac{r+1}{r+2} - \frac{r}{r+1} \\ &= \frac{(r+1)^2 - r(r+2)}{(r+1)(r+2)} \\ &= \frac{r^2 + 2r + 1 - r^2 - 2r}{(r+1)(r+2)} \\ &= \frac{1}{(r+1)(r+2)} \\ &= \text{RHS, as required}\end{aligned}$$

To show that an algebraic identity is true, you should start from one side of the identity, here the left hand side (LHS), and use algebra to show that it is equal to the other side of the identity, here the right hand side (RHS).

You use the identity that you proved in part **a** to break up each term in the summation into two parts.

This is the LHS of the identity with $r = 1$.

This is the LHS of the identity with $r = 2$.

This is the LHS of the identity with $r = 3$.

b $$\sum_{r=1}^{n}\frac{1}{(r+1)(r+2)}=\sum_{r=1}^{n}\left(\frac{r+1}{r+2}-\frac{r}{r+1}\right)$$

$$\begin{aligned}
&=\cancel{\frac{2}{3}}-\frac{1}{2}\\
&\quad+\cancel{\frac{3}{4}}-\cancel{\frac{2}{3}}\\
&\quad+\cancel{\frac{4}{5}}-\cancel{\frac{3}{4}}\\
&\quad\vdots\\
&\quad+\cancel{\frac{n}{n+1}}-\cancel{\frac{n-1}{n}}\\
&\quad+\frac{n+1}{n+2}-\cancel{\frac{n}{n+1}}\\
&=\frac{n+1}{n+2}-\frac{1}{2}\\
&=\frac{2(n+1)-(n+2)}{2(n+2)}=\frac{2n+2-n-2}{2(n+2)}\\
&=\frac{n}{2(n+2)}
\end{aligned}$$

This is the LHS of the identity with $r = n - 1$.

$$\frac{r+1}{r+2}-\frac{r}{r+1}=\frac{n-1+1}{n-1+2}-\frac{n-1}{n-1+1}$$
$$=\frac{n}{n+1}-\frac{n-1}{n}$$

This is the LHS of the identity with $r = n$.

The only terms which have not cancelled one another out are the $-\frac{1}{2}$ in the first line of the summation and the $\frac{n+1}{n+2}$ in the last line.

21 a Let $\frac{2}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$

Multiplying throughout by $(x+1)(x+2)(x+3)$

$2 = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$

When -1 is substituted for x then both $B(x+1)(x+3)$ and $C(x+1)(x+2)$ become

Substitute $x=-1$

$2 = A\times 1\times 2 \Rightarrow A = 1$

Substitute $x=-2$

$2 = B\times -1\times 1 \Rightarrow B = -2$

Substitute $x=-3$

$2 = C\times -2\times -1 \Rightarrow C = 1$

Hence

$$\mathrm{f}(x) = \frac{1}{x+1} - \frac{2}{x+2} + \frac{1}{x+3}$$

You use the partial fractions in part **a** to break up each term in the summation into three parts.

b Using the result in part **a** with $x = r$

$$\sum_{r=1}^{n} \mathrm{f}(r) = \frac{1}{r+1} - \frac{2}{r+2} + \frac{1}{r+3}$$

$$= \frac{1}{2} - \frac{2}{3} + \cancel{\frac{1}{4}}$$

$$+ \frac{1}{3} - \cancel{\frac{2}{4}} + \cancel{\frac{1}{5}}$$

$$+ \cancel{\frac{1}{4}} - \cancel{\frac{2}{5}} + \cancel{\frac{1}{6}}$$

$$\vdots$$

$$+ \cancel{\frac{1}{n-1}} - \cancel{\frac{2}{n}} + \cancel{\frac{1}{n+1}}$$

$$+ \cancel{\frac{1}{n}} - \cancel{\frac{2}{n+1}} + \frac{1}{n+2}$$

$$+ \cancel{\frac{1}{n+1}} - \frac{2}{n+2} + \frac{1}{n+3}$$

Three terms at the beginning of the summation and three terms at the end have not been cancelled out.

$$= \frac{1}{2} - \frac{2}{3} + \frac{1}{3} + \frac{1}{n+2} - \frac{2}{n+2} + \frac{1}{n+3}$$

$$= \frac{1}{6} - \frac{1}{n+2} + \frac{1}{n+3}$$

This question asks for no particular form of the answer. You should collect together like terms but, otherwise, the expression can be left as it is. You do not have to express your answer as a single fraction unless the question asks you to do this.

22 a $$\frac{1}{(r-1)^2}-\frac{1}{r^2}=\frac{r^2-(r-1)^2}{r^2(r-1)^2}$$
$$=\frac{r^2-(r^2-2r+1)}{r^2(r-1)^2}$$
$$=\frac{2r-1}{r^2(r-1)^2}$$

Methods for simplifying algebraic fractions can be found in Chapter 1 of book C3.

b $$\sum_{r=2}^{n}\frac{2r-1}{r^2(r-1)^2}=\sum_{r=2}^{n}\left(\frac{1}{(r-1)^2}-\frac{1}{r^2}\right)$$
$$=\frac{1}{1^2}-\cancel{\frac{1}{2^2}}$$
$$+\cancel{\frac{1}{2^2}}-\cancel{\frac{1}{3^2}}$$
$$+\cancel{\frac{1}{3^2}}-\cancel{\frac{1}{4^2}}$$
$$\vdots$$
$$+\cancel{\frac{1}{(n-2)^2}}-\cancel{\frac{1}{(n-1)^2}}$$
$$+\cancel{\frac{1}{(n-1)^2}}-\frac{1}{n^2}$$
$$=\frac{1}{1^2}-\frac{1}{n^2}=1-\frac{1}{n^2}\text{, as required}$$

This summation starts from $r=2$ and not from the more common $r=1$. It could not start from $r=1$ as $\dfrac{1}{(r-1)^2}$ is not defined for that value.

In this summation all of the terms cancel out with one another except for one term at the beginning and one term at the end.

23 a

$$\frac{4}{r(r+2)}=\frac{A}{r}+\frac{B}{r+2}$$
$$4=A(r+2)+Br$$
$$4=2A=-2B$$

Let $\mathrm{f}(r)=\dfrac{1}{r}$

$$\sum_{r=1}^{n}\frac{4}{r(r+2)}=2\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+2}\right)$$
$$=2(\mathrm{f}(1)+\mathrm{f}(2)-\mathrm{f}(n+1)-\mathrm{f}(n+2))$$
$$=2\left(1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right)$$
$$=\frac{3(n+1)(n+2)-2(n+2)-2(n+1)}{(n+1)(n+2)}$$
$$=\frac{n(3n+5)}{(n+1)(n+2)}$$

Hence $a=3$ and $b=5$

23 b $\displaystyle\sum_{r=50}^{100}\frac{4}{r(r+2)}=\sum_{r=1}^{100}\frac{4}{r(r+2)}-\sum_{r=1}^{49}\frac{4}{r(r+2)}$

$$=\frac{100\times305}{101\times102}-\frac{49\times152}{50\times51}$$

$$=2.960\ 590\ldots-2.920\ 784$$

$$=0.0398\ (4\text{ d.p.})$$

$\displaystyle\sum_{r=50}^{100}f(r)=\sum_{r=1}^{100}f(r)-\sum_{r=1}^{49}f(r)$

You find the sum from the 50th to the 100th term by subtracting the sum from the first to the 49th term from the sum from the first to the 100th term.

It is a common error to subtract one term too many, in this case the 50th term. The sum you are finding starts with the 50th term. You must not subtract it from the series – you have to leave it in the series.

24 a $4r^2-1=(2r-1)(2r+1)$

Let

$$\frac{2}{4r^2-1}=\frac{2}{(2r-1)(2r+1)}=\frac{A}{2r-1}+\frac{B}{2r+1}$$

Multiply throughout by $(2r-1)(2r+1)$

$$2=A(2r+1)+B(2r-1)$$

Substitute $r=\frac{1}{2}$

$$2=2A\Rightarrow A=1$$

Substitute $r=-\frac{1}{2}$

$$2=-2B\Rightarrow B=-1$$

Hence

$$\frac{2}{4r^2-1}=\frac{1}{2r-1}-\frac{1}{2r+1}$$

$$\sum_{r=1}^{n}\frac{2}{4r^2-1}=\sum_{r=1}^{n}\left(\frac{1}{2r-1}-\frac{1}{2r+1}\right)$$

$$=\frac{1}{1}-\cancel{\frac{1}{3}}$$

$$+\cancel{\frac{1}{3}}-\cancel{\frac{1}{5}}$$

$$+\cancel{\frac{1}{5}}-\cancel{\frac{1}{7}}$$

$$\vdots$$

$$+\cancel{\frac{1}{2n-3}}-\cancel{\frac{1}{2n-1}}$$

$$+\cancel{\frac{1}{2n-1}}-\frac{1}{2n+1}$$

$$=1-\frac{1}{2n+1},\text{ as required}$$

This question gives you the option to choose your own method (the question has 'or otherwise') and, as you are given the answer, you could, if you preferred, use the method of mathematical induction.

If the method of differences is used, you begin by factorising $4r^2-1$, using the difference of two squares, and then express $\dfrac{2}{(2r-1)(2r+1)}$ in partial fractions.

With $r=1$,

$$\frac{1}{2r-1}-\frac{1}{2r+1}=\frac{1}{2\times1-1}-\frac{1}{2\times1+1}=\frac{1}{1}-\frac{1}{3}$$

With $r=n-1$,

$$\frac{1}{2r-1}-\frac{1}{2r+1}=\frac{1}{2\times(n-1)-1}-\frac{1}{2\times(n-1)}$$

$$=\frac{1}{2n-2-1}-\frac{1}{2n-2+1}=\frac{1}{2n-3}-\frac{1}{2n-1}$$

The only terms which are not cancelled out in the summation are the $\frac{1}{1}$ at the beginning and the $-\frac{1}{2n+1}$ at the end.

24 b $$\sum_{r=11}^{n}\frac{2}{4r^2-1}=\sum_{r=1}^{20}\frac{2}{4r^2-1}-\sum_{r=1}^{10}\frac{2}{4r^2-1}$$

$$=\left(1-\frac{1}{41}-1+\frac{1}{21}\right)$$

$$=-\frac{1}{41}+\frac{1}{21}=\frac{-21+41}{41\times 21}$$

$$=\frac{20}{861}$$

You find the sum from the 11th to the 20th term by subtracting the sum from the first to the 10th term from the sum from the first to the 20th term.

The conditions of the question require an exact answer, so you must not use decimals.

25 a Using the binomial expansion

$$(2r+1)^3=8r^3+12r^2+6r+1 \qquad \textbf{(1)}$$

$$(2r-1)^3=8r^3-12r^2+6r-1 \qquad \textbf{(2)}$$

Subtracting **(2)** from **(1)**

$$(2r+1)^3-(2r-1)^3=24r^2+2 \qquad \textbf{(3)}$$

$A=24,\ B=2$

Subtracting the two expansions gives an expression in r^2. This enables you to sum r^2 using the method of differences.

b Using identity **(3)** in part **a**

$$\sum_{r=1}^{n}(24r^2+2)=\sum_{r=1}^{n}\left((2r+1)^3-(2r-1)^3\right)$$

$$24\sum_{r=1}^{n}r^2+\sum_{r=1}^{n}2=\sum_{r=1}^{n}\left((2r+1)^3-(2r-1)^3\right)$$

$$\sum_{r=1}^{n}2=\underbrace{2+2+2+\ldots+2}_{n\text{ times}}=2n$$

It is a common error to write $\sum_{r=1}^{n}2=2$.

$$24\sum_{r=1}^{n}r^2+2n=\cancel{3^3}-1^3$$
$$+\cancel{5^3}-\cancel{3^3}$$
$$+\cancel{7^3}-\cancel{5^3}$$
$$\vdots$$
$$+\cancel{(2n-1)^3}-\cancel{(2n-3)^3}$$
$$+(2n+1)^3-\cancel{(2n-1)^3}$$

The expression is $(2r+1)^3-(2r-1)^3$ with $n-1$ substituted for r.

$$(2(n-1)+1)^3-(2(n-1)-1)^3$$
$$=(2n-1)^3-(2n-3)^3$$

$$24\sum_{r=1}^{n}r^2+2n=(2n+1)^3-1$$

$$24\sum_{r=1}^{n}r^2=8n^3+12n^2+6n+1-1-2n$$

$$=8n^3+12n^2+4n=4n(2n^2+3n+1)$$

$$=4n(n+1)(2n+1)$$

$$\sum_{r=1}^{n}r^2=\frac{4n(n+1)(2n+1)}{24}=\tfrac{1}{6}n(n+1)(2n+1),\text{ as required.}$$

Summing gives you an equation in $\sum r^2$, which you solve. You then factorise the result to give the answer in the form required by the question.

25 c $(3r-1)^2 = 9r^2 - 6r + 1$

Hence

$$\sum_{r=1}^{40}(3r-1)^2 = 9\sum_{r=1}^{40}r^2 - 6\sum_{r=1}^{40}r + \sum_{r=1}^{40}1$$

In the formula proved in part **b**, you replace the n by 40.

Using the result in part **b**.

$$9\sum_{r=1}^{40}r^2 = 9\times\tfrac{1}{6}\times 40\times 41\times 81 = 199\ 260$$

Using the standard result $\sum_{r=1}^{n} r = \dfrac{n(n+1)}{2}$,

$$6\sum_{r=1}^{40} r = 6\times\frac{40\times 41}{2} = 4920$$

$$\sum_{r=1}^{40}1 = 40$$

$\sum_{r=1}^{40}1 = 40$ is 40 ones added together which is, of course, 40.

Combining these results

$$\sum_{r=1}^{40}(3r-1)^2 = 199\ 260 - 4920 + 40 = 194\ 380$$

26 $$\frac{1}{r(r+1)(r+2)} = \frac{A}{r} + \frac{B}{r+1} + \frac{C}{r+2}$$

$1 = A(r+1)(r+2) + Br(r+2) + Cr(r+1)$

$r = 0: 1 = 2A$

$r = 1: 1 = -B$

$r = 2: 1 = 2C$

Let $f(r) = \dfrac{1}{r}$

$$\begin{aligned}\sum_{r=1}^{2n}\frac{1}{r(r+1)(r+2)} &= \frac{1}{2}\sum_{r=1}^{2n}\left(\frac{1}{r} - \frac{1}{r+1} + \frac{1}{r+2} - \frac{1}{r+1}\right)\\
&= \frac{1}{2}(f(1) - f(2n+1) + f(2n+2) - f(2))\\
&= \frac{1}{2}\left(1 - \frac{1}{2} - \frac{1}{2n+1} + \frac{1}{2n+2}\right)\\
&= \frac{1}{4}\left(1 - \frac{2}{2n+1} + \frac{1}{n+1}\right)\\
&= \frac{1}{4}\left(\frac{(n+1)(2n+1) - 2(n+1) + (2n+1)}{(n+1)(2n+1)}\right)\\
&= \frac{1}{4}\,\frac{2n^2+3n}{(n+1)(2n+1)} = \frac{n(2n+3)}{4(n+1)(2n+1)}\end{aligned}$$

Hence $a = 2, b = 3$ and $c = 4$

27 a RHS $= r - 1 + \frac{1}{r} - \frac{1}{r+1}$

$= \frac{(r-1)r(r+1) + (r+1) - r}{r(r+1)}$

$= \frac{r(r^2-1)+1}{r(r+1)}$

$= \frac{r^3 - r + 1}{r(r+1)} =$ LHS, as required

To show that an algebraic identity is true, you should start from one side of the identity, here the right hand side (RHS), and use algebra to show that it is equal to the other side of the identity, here the left hand side (LHS).

b Using the result in part **a**

This summation is broken up into 3 separate summations. Only the third of these uses the method of differences.

$$\sum_{r=1}^{n} \frac{r^3 - r + 1}{r(r+1)} = \sum_{r=1}^{n} r - \sum_{r=1}^{n} 1 + \sum_{r=1}^{n} \left(\frac{1}{r} - \frac{1}{r+1}\right)$$

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

$$\sum_{r=1}^{n} 1 = n$$

$$\sum_{r=1}^{n} \left(\frac{1}{r} - \frac{1}{r+1}\right) = \frac{1}{1} - \cancel{\frac{1}{2}}$$
$$+ \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}$$
$$+ \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}$$
$$\vdots$$
$$+ \cancel{\frac{1}{n-1}} + \cancel{\frac{1}{n}}$$
$$+ \cancel{\frac{1}{n}} - \frac{1}{n+1}$$
$$= 1 + \frac{1}{n+1}$$

In the summation, using the method of differences, all of the terms cancel out with one another except for one term at the beginning and one term at the end.

Combining the three summations

$$\sum_{r=1}^{n} \frac{r^3 - r + 1}{r(r+1)} = \frac{n(n+1)}{2} - n + 1 - \frac{1}{n+1}$$

$$= \frac{n(n+1)^2 - 2n(n+1) + 2(n+1) - 2}{2(n+1)}$$

$$= \frac{n^3 + 2n^2 + n - 2n^2 - 2n + 2n + 2 - 2}{2(n+1)}$$

$$= \frac{n^3 + n}{2(n+1)} = \frac{n(n^2+1)}{2(n+1)}$$

To complete the question, you put the results of the three summations over a common denominator and simplify the resulting expression as far as possible.

28 $\frac{2r+3}{r(r+1)}=\frac{A}{r}+\frac{B}{r+1}$

$2r+3=A(r+1)+Br$

$3=A, 1=-B$

$$\frac{2r+3}{r(r+1)}\frac{1}{3^r}=\frac{1}{3^r}\left(\frac{3}{r}-\frac{1}{r+1}\right)$$

$$=\frac{1}{3^{r-1}}\frac{1}{r}-\frac{1}{3^r}\frac{1}{r+1}$$

Let $\mathrm{f}(r)=\frac{1}{3^{r-1}}\frac{1}{r}$

$$\sum_{r=1}^{n}\frac{2r+3}{r(r+1)}\frac{1}{3^r}=\sum_{r=1}^{n}\left(\mathrm{f}(r)-\mathrm{f}(r+1)\right)$$

$$=(\mathrm{f}(1)-\mathrm{f}(n+1))$$

$$=1-\frac{1}{3^n(n+1)}$$

29 Using Euler's solution $\mathrm{e}^{\mathrm{i}\theta}=\cos\theta+\mathrm{i}\sin\theta$,

$$\cos 2x+\mathrm{i}\sin 2x=\mathrm{e}^{\mathrm{i}2x}$$

$$\cos 9x-\mathrm{i}\sin 9x=\cos(-9x)+\mathrm{i}\sin(-9x)=\mathrm{e}^{\mathrm{i}(-9x)}$$

For any angle, $\theta, \cos\theta=\cos(-\theta)$ and $-\sin\theta=\sin(-\theta)$
You will find these relations useful when finding the arguments of

Hence

$$\frac{\cos 2x+\mathrm{i}\sin 2x}{\cos 9x-\mathrm{i}\sin 9x}=\frac{\mathrm{e}^{\mathrm{i}2x}}{\mathrm{e}^{\mathrm{i}(-9x)}}=\mathrm{e}^{\mathrm{i}(2x+9x)}=\mathrm{e}^{\mathrm{i}11x}$$

Manipulating the arguments in $\mathrm{e}^{\mathrm{i}\theta}$ you use the ordinary laws of indices.

$$=\cos 11x+\mathrm{i}\sin 11x$$

This is the required form with $n=11$.

30 a By de Moivre's theorem

It is sensible to abbreviate $\cos\theta$ and $\sin\theta$ as c and s respectively when you have as many powers of $\cos\theta$ and $\sin\theta$ to write out as you have

$$\cos 5\theta+\mathrm{i}\sin 5\theta=(\cos\theta+\mathrm{i}\sin\theta)^5=(c+\mathrm{i}s)^5\text{, say}$$

$$=c^5+5c^4\mathrm{i}s+10c^3\mathrm{i}^2s^2+10c^2\mathrm{i}^3s^3+5c\mathrm{i}^4s^4+\mathrm{i}^5s^5$$

$$=c^5+\mathrm{i}5c^4s-10c^3s^3-\mathrm{i}10c^2s^3+5cs^4-\mathrm{i}s^5$$

Use the binomial expansion.

Use $\mathrm{i}^2=-1, \mathrm{i}^3=-\mathrm{i}, \mathrm{i}^4=1$

Equating real parts

$$\cos 5\theta=c^5-10c^3s^2+5cs^4$$

Using $\cos^2\theta+\sin^2\theta=1$

$$\cos 5\theta=c^5-10c^3(1-c^2)+5c(1-c^2)^2$$

$$=c^5-10c^3+10c^5+5c-10c^3+5c^5$$

$$=16c^5-20c^3+5c$$

$$=16\cos^5\theta-20\cos^3\theta+5\cos\theta\text{, as required}$$

30 b Substitute $x = \cos\theta$ into $16x^5 - 20x^3 + 5x + 1 = 0$

$$16\cos^5\theta - 20\cos^3\theta + 5\cos\theta + 1 = 0$$
$$16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = -1$$

Using the result of part **a**

$$\cos 5\theta = -1$$
$$5\theta = \pi,\ 3\pi,\ 5\pi$$
$$\theta = \frac{\pi}{5},\ \frac{3\pi}{5},\ \frac{5\pi}{5}$$

Additional solutions are found by increasing the angles in steps of 2π. You are asked for 3 answers, so you need 3 angles at this stage.

$$x = \cos\theta = \cos\frac{\pi}{5},\ \cos\frac{3\pi}{5},\ \cos\pi$$
$$= 0.809,\ -0.309,\ -1$$

The two approximate answers are given to 3 decimal places, as the question specified; the remaining answer -1 is exact.

31 a By de Moivre's theorem

$$\cos 5\theta + \mathrm{i}\sin 5\theta = (\cos\theta + \mathrm{i}\sin\theta)^5 = (c + \mathrm{i}s)^5, \text{ say}$$
$$= c^5 + 5c^4\mathrm{i}s + 10c^3\mathrm{i}^2s^2 + 10c^2\mathrm{i}^2s^3 + 5c\mathrm{i}^3s^4 + \mathrm{i}^5s^5$$
$$= c^5 + \mathrm{i}5c^4s - 10c^3s^2 - \mathrm{i}10c^2s^3 + 5cs^4 - \mathrm{i}s^5$$

Equating imaginary parts

$$\sin 5\theta = 5c^4s - 10c^2s^3 + s^5$$
$$= s(5c^4 - 10c^2s^2 + s^4)$$
$$= s(5c^4 - 10c^2(1-c^2) + (1-c^2)^2)$$
$$= s(5c^4 - 10c^2 + 10c^4 + 1 - 2c^2 + c^4)$$
$$= s(16c^4 - 12c^2 + 1)$$
$$= \sin\theta(16\cos^4\theta - 12\cos^2\theta + 1), \text{ as required}$$

Repeatedly using the identity $\cos^2\theta + \sin^2\theta = 1$, which in this context is $s^2 = 1 - c^2$.

31 b

$$\sin 5\theta + \cos\theta \sin 2\theta = 0$$

$$\sin\theta(16\cos^4\theta - 12\cos^2\theta + 1) + 2\sin\theta\cos^2\theta = 0$$

Using the identity proved in part **a** and the identity $\sin 2\theta = 2\sin\theta\cos\theta$.

$$\sin\theta(16\cos^4\theta - 10\cos^2\theta + 1) = 0$$

$$\sin\theta(2\cos^2\theta - 1)(8\cos^2\theta - 1) = 0$$

Hence $\sin\theta = 0,\ \cos^2\theta = \frac{1}{2},\ \cos^2\theta = \frac{1}{8}$

$$\sin\theta = 0 \Rightarrow \theta = 0$$

$$\cos^2\theta = \frac{1}{2} \Rightarrow \cos\theta = \pm\frac{1}{\sqrt{2}}$$

$$\cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$$

$$\cos\theta = -\frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{3\pi}{4}$$

You must consider the negative as well as the positive square roots.

$$\cos^2\theta = \frac{1}{8} \Rightarrow \cos\theta = \pm\frac{1}{2\sqrt{2}}$$

$$\cos\theta = \frac{1}{2\sqrt{2}} \Rightarrow \theta = 1.209 \text{ (3 d.p.)}$$

$$\cos\theta = -\frac{1}{2\sqrt{2}} \Rightarrow \theta = 1.932 \text{ (3 d.p.)}$$

The question has specified no accuracy and any sensible accuracy would be accepted for the approximate answers.

The solutions of the equation are

$0,\ \frac{\pi}{4},\ \frac{3\pi}{4},\ 1.209$ (3 d.p.) and 1.932 (3 d.p.).

32 a $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

Let $z = e^{i\theta}$

Putting $z = e^{i\theta}$ shortens the working.

then $\sin\theta = \dfrac{z - z^{-1}}{2i}$

$$\sin^5\theta = \left(\frac{z - z^{-1}}{2i}\right)^5$$

Use Pascal's triangle to remember the coefficients in $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$.

$$= \frac{1}{(2i)^5}(z^5 - 5z^4 \times z^{-1} + 10z^3 \times z^{-2} - 10z^2 \times z^{-3} + 5z \times z^{-4} - z^{-5})$$

$$= \frac{1}{32i}(z^5 - 5z^3 + 10z - 10z^{-1} + 5z^{-3} - z^{-5})$$

$$= \frac{1}{16}\left(\frac{z^5 - z^{-5}}{2i} - \frac{5(z^3 - z^{-3})}{2i} + \frac{10(z - z^{-1})}{2i}\right)$$

The general relation is $\sin n\theta = \dfrac{e^{in\theta} - e^{-in\theta}}{2i} = \dfrac{z^n - z^{-n}}{2i}$

$$= \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta), \text{ as required}$$

b $\displaystyle\int_0^{\frac{\pi}{2}} \sin^5\theta \, d\theta = \frac{1}{16}\int_0^{\frac{\pi}{2}} (\sin 5\theta - 5\sin 3\theta + 10\sin\theta)\, d\theta$

Each term on the right hand side of the identity shown in part **a** can be integrated using the formula $\displaystyle\int \sin n\theta \, d\theta = -\frac{\cos n\theta}{n}$.

$$= \frac{1}{16}\left[-\frac{1}{5}\cos 5\theta + \frac{5}{3}\cos 3\theta - 10\cos\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{16}\left(0 - \left(-\frac{1}{5} + \frac{5}{3} - 10\right)\right)$$

$$= \frac{1}{16} \times \frac{128}{15} = \frac{8}{15}, \text{ as required}$$

33 a $z = \cos\theta + i\sin\theta$

Using de Moivre's theorem

$z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ **(1)**

From **(1)**

$$z^{-n} = \frac{1}{z^n} = \frac{1}{\cos n\theta + i\sin n\theta}$$

$$= \frac{1}{\cos n\theta + i\sin n\theta} \times \frac{\cos n\theta - i\sin n\theta}{\cos n\theta - i\sin n\theta}$$

Multiply the numerator and denominator by $\cos n\theta - i\sin n\theta$, the conjugate complex number of $\cos n\theta + i\sin n\theta$.

$$= \frac{\cos n\theta - i\sin n\theta}{\cos^2 n\theta + \sin^2 n\theta} = \cos n\theta - i\sin n\theta \quad \textbf{(2)}$$

Use $\cos^2 n\theta + \sin^2 n\theta = 1$.

$z^n + z^{-n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$, as required.

33 b $\cos\theta = \dfrac{e^{i\theta} + e^{i\theta}}{2} = \dfrac{z + z^{-1}}{2}$

$$\cos^6\theta = \left(\frac{z + z^{-1}}{2}\right)^6$$

$$= \frac{1}{64}\left(z^6 + 6z^5z^{-1} + 15z^4z^{-2} + 20z^3z^{-3} + 15z^2z^{-4} + 6z^1z^{-5} + z^{-6}\right)$$

Pair the terms as shown.

$$= \frac{1}{64}\left(z^6 + 6z^4 + 15z^2 + 20 + 15z^{-2} + 6z^{-4} + z^{-6}\right)$$

$$= \frac{1}{32}\left(\frac{z^6 + z^{-6}}{2} + \frac{6(z^4 + z^{-4})}{2} + \frac{15(z^2 + z^{-2})}{2} + \frac{20}{2}\right)$$

$$= \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$$

You use $\dfrac{z^n + z^{-n}}{2} = \cos n\theta$ with $n = 6,\ 4$ and 2.

c $\displaystyle\int_0^{\frac{\pi}{2}} \cos^6\theta \,d\theta = \frac{1}{32}\int_0^{\frac{\pi}{2}} (\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)\,d\theta$

$$= \frac{1}{32} - \left[\frac{1}{6}\sin 6\theta + \frac{6}{4}\sin 4\theta + \frac{15}{2}\sin 2\theta + 10\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{32} \times 10 \times \frac{\pi}{2} = \frac{5\pi}{32}, \text{ as required}$$

With the exception of 10θ all of these terms have value 0 at both the upper and the lower limit.

34 a Let $4+4\mathrm{i}=r(\cos\theta+\mathrm{i}\sin\theta)=r\cos\theta+\mathrm{i}r\sin\theta$

Equating real parts

$4=r\cos\theta$ **(1)**

Equating imaginary parts

$4=r\sin\theta$ **(2)**

Dividing **(2)** by **(1)**

$\tan\theta=1\Rightarrow\theta=\frac{\pi}{4}$

Substituting $\theta=\frac{\pi}{4}$ into **(1)**

$4=r\cos\frac{\pi}{4}\Rightarrow 4=r\times\frac{1}{\sqrt{2}}\Rightarrow r=4\sqrt{2}$

Hence

$4+4\mathrm{i}=4\sqrt{2}\left(\cos\frac{\pi}{4}+\mathrm{i}\sin\frac{\pi}{4}\right)$

$=2^{\frac{5}{2}}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$

$z^5=2^{\frac{5}{2}}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}},\ 2^{\frac{5}{2}}\mathrm{e}^{\mathrm{i}\frac{9\pi}{4}},2^{\frac{5}{2}}\mathrm{e}^{\mathrm{i}\frac{17\pi}{4}},\ 2^{\frac{5}{2}}\mathrm{e}^{\mathrm{i}\frac{25\pi}{4}},\ 2^{\frac{5}{2}}\mathrm{e}^{\mathrm{i}\frac{33\pi}{4}}$

$z=2^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\frac{\pi}{20}},\ 2^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\frac{9\pi}{20}},\ 2^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\frac{17\pi}{20}},\ 2^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\frac{25\pi}{20}},\ 2^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\frac{33\pi}{20}}$

This is the required form with $r=\sqrt{2}$ and

$k=\frac{1}{20},\frac{9}{20},\frac{17}{20},\frac{25}{20}\left(=\frac{5}{4}\right),\frac{33}{20}.$

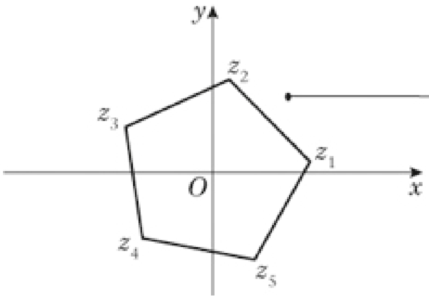
Finding the roots of a complex number is usually easier if you obtain the number in the form $r\mathrm{e}^{\mathrm{i}\theta}$. As you will use Euler's relation, the first step towards this is to get the complex number into the form $r(\cos\theta+\mathrm{i}\sin\theta)$.

To take the fifth root, write $4\sqrt{2}=2^{\frac{5}{2}}$.

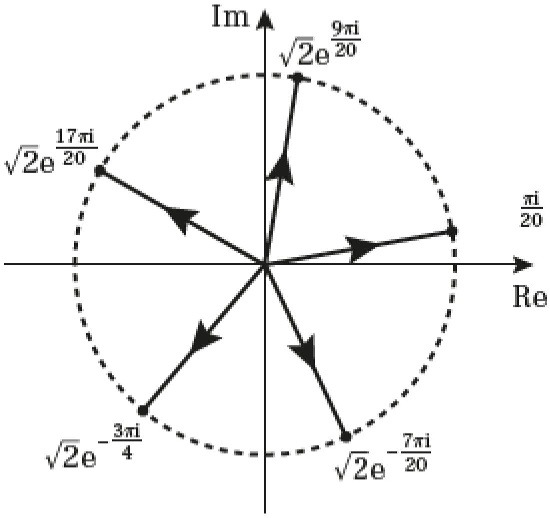
For example, if $z^5=2^{\frac{5}{2}}\mathrm{e}^{\mathrm{i}\frac{9\pi}{4}}$ then

$z=\left(2^{\frac{5}{2}}\mathrm{e}^{\mathrm{i}\frac{9\pi}{4}}\right)^{\frac{1}{5}}=2^{\frac{5}{2}\times\frac{1}{5}}\mathrm{e}^{\mathrm{i}\frac{9\pi}{4}\times\frac{1}{5}}=2^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\frac{9\pi}{20}}.$

34 b



The points representing the 5 roots are the vertices of a regular pentagon.



35 a Let $32+32\sqrt{3}\mathrm{i}=r(\cos\theta+\mathrm{i}\sin\theta)=r\cos\theta+\mathrm{i}r\sin\theta$

Equating real parts

$32=r\cos\theta$ **(1)**

Equating imaginary parts

$32\sqrt{3}=r\sin\theta$ **(2)**

Dividing **(2)** by **(1)**

$\tan\theta=\sqrt{3}\Rightarrow\theta=\frac{\pi}{3}$

Substituting $\theta=\frac{\pi}{3}$ into **(1)**

$32=r\cos\frac{\pi}{3}\Rightarrow 32=r\times\frac{1}{2}\Rightarrow r=64$

Hence

$32+32\sqrt{3}\mathrm{i}=64\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right)$

$=64\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$

Finding the roots of a complex number is usually easier if you obtain the number in the form $r\mathrm{e}^{\mathrm{i}\theta}$. As you will use Euler's relation, the first step towards this is to get the complex number into the form $r(\cos\theta+\mathrm{i}\sin\theta)$.

$z^3=64\mathrm{e}^{\mathrm{i}\frac{\pi}{3}},\ 64\mathrm{e}^{\mathrm{i}\frac{7\pi}{3}},\ 64\mathrm{e}^{\mathrm{i}\frac{-5\pi}{3}}$

$z=4\mathrm{e}^{\mathrm{i}\frac{\pi}{9}},\ 4\mathrm{e}^{\mathrm{i}\frac{7\pi}{9}},\ 4\mathrm{e}^{-\mathrm{i}\frac{5\pi}{9}}$

The solutions are $r\mathrm{e}^{\mathrm{i}\theta}$ where $r=4$ and

$\theta=-\frac{5\pi}{9},\frac{\pi}{9},\frac{7\pi}{9}$

Additional solutions are found by increasing or decreasing the arguments in steps of 2π. You area asked for 3 answers, so you need 3 arguments. Had you increased the argument $\frac{7\pi}{3}$ by 2π to $\frac{13\pi}{3}$, this would have given a correct solution to the equation but it would lead to $\theta=\frac{13\pi}{9}$, which does not satisfy the condition $\theta\le\pi$ in the question. So the third argument has to be found by subtracting 2π from $\frac{\pi}{3}$.

b $z=4\mathrm{e}^{\mathrm{i}\frac{\pi}{9}},\ 4\mathrm{e}^{\mathrm{i}\frac{7\pi}{9}},\ 4\mathrm{e}^{-\mathrm{i}\frac{5\pi}{9}}$

$z^9=\left(4\mathrm{e}^{\mathrm{i}\frac{\pi}{9}}\right)^9,\ \left(4\mathrm{e}^{\mathrm{i}\frac{7\pi}{9}}\right)^9,\ \left(4\mathrm{e}^{-\mathrm{i}\frac{5\pi}{9}}\right)^9$

$=4^9\mathrm{e}^{\mathrm{i}\pi},\ 4^9\mathrm{e}^{\mathrm{i}7\pi},\ 4^9\mathrm{e}^{-\mathrm{i}5\pi}$

$\mathrm{e}^{\mathrm{i}\pi}=\cos\pi+\mathrm{i}\sin\pi=-1+0=-1$. Similarly for the arguments 7π and -5π.

The value of all three of these expressions is $-4^9=-2^{18}$

Hence the solutions satisfy $z^9+2^k=0$, where $k=18$.

36 $\mathrm{i} = \cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2} = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$

The modulus of the complex number i is 1 and its argument is $\frac{\pi}{2}$. So $\mathrm{i} = 1\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$.

$z^5 = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}, \mathrm{e}^{\mathrm{i}\frac{5\pi}{2}}, \mathrm{e}^{\mathrm{i}\frac{9\pi}{2}}, \mathrm{e}^{\mathrm{i}\frac{13\pi}{2}}, \mathrm{e}^{\mathrm{i}\frac{17\pi}{2}},$

Additional solutions are found by increasing the arguments in steps of 2π. As the equation is of degree 5, there are exactly 5 distinct answers.

$z = \mathrm{e}^{\mathrm{i}\frac{\pi}{10}}, \mathrm{e}^{\mathrm{i}\frac{5\pi}{10}}, \mathrm{e}^{\mathrm{i}\frac{9\pi}{10}}, \mathrm{e}^{\mathrm{i}\frac{13\pi}{10}}, \mathrm{e}^{\mathrm{i}\frac{17\pi}{10}}$

For example, if $z^5 = \mathrm{e}^{\mathrm{i}\frac{9\pi}{2}}$ then $z = \left(\mathrm{e}^{\mathrm{i}\frac{9\pi}{2}}\right)^{\frac{1}{5}} = \left(\mathrm{e}^{\mathrm{i}\frac{9\pi}{10}}\right)$.

Hence

$z = \cos\theta + \mathrm{i}\sin\theta$, where

$\theta = \frac{\pi}{10}, \frac{5\pi}{10}\left(=\frac{\pi}{2}\right), \frac{9\pi}{10}, \frac{13\pi}{10}, \frac{17\pi}{10}$

37 a $z^5 = 16 + 16i\sqrt{3} = 32\left(\cos\left(\frac{\pi}{3} + 2k\pi\right) + i\sin\left(\frac{\pi}{3} + 2k\pi\right)\right)$

as $\sqrt{16^2 + \left(16\sqrt{3}\right)^2} = 32, \arctan\frac{16\sqrt{3}}{16} = \frac{\pi}{3}$

$z = r(\cos\theta + i\sin\theta)$

$z^5 = r^5(\cos 5\theta + i\sin 5\theta)$

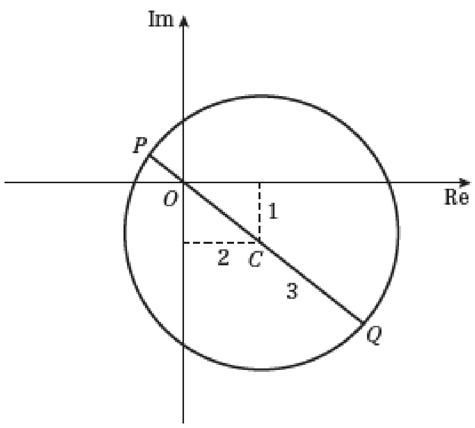
$r^5 = 32, r = 2$

$\theta = \frac{\pi}{15} + \frac{2k\pi}{5} = \frac{\pi}{15}, \frac{7\pi}{15}, \frac{13\pi}{15}, -\frac{\pi}{3}, -\frac{11\pi}{15}$

$z = 2e^{\frac{i\pi}{15}}, 2e^{\frac{7i\pi}{15}}, 2e^{\frac{13i\pi}{15}}, 2e^{-\frac{i\pi}{3}}, 2e^{-\frac{11i\pi}{15}}$

b The polygon is a regular pentagon.

38 a $|z - 2 + \mathrm{i}| = 3$ is the circle with centre (2, −1) and radius 3

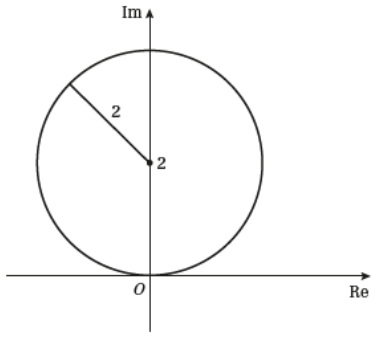


b $|OC| = \sqrt{1^2 + 2^2} = \sqrt{5}$

Therefore,

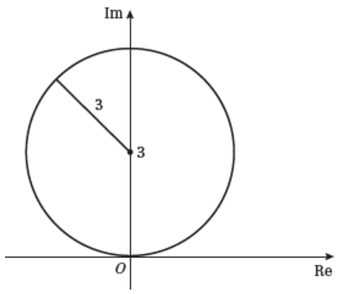
Min $|z| = 3 - \sqrt{5}$ and Max $|z| = 3 + \sqrt{5}$

39 a $|z-2\text{i}|=2$ is the circle with centre (0, 2) and radius 2

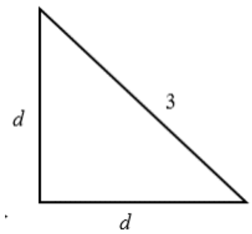


b $|z|_{\max}=4$

40 a $|z-3\text{i}|=3$ is the circle with centre (0, 3) and radius 3



b $\arg(z-3\text{i})=\frac{3\pi}{4}$ is the half-line originating at (0, 3) at $\frac{3\pi}{4}$ to the positive x-axis

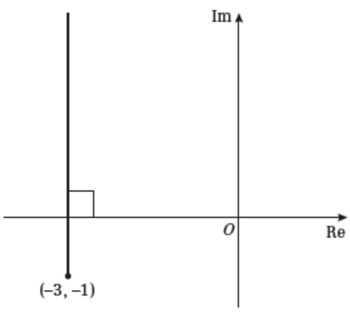


$2d^2=9$

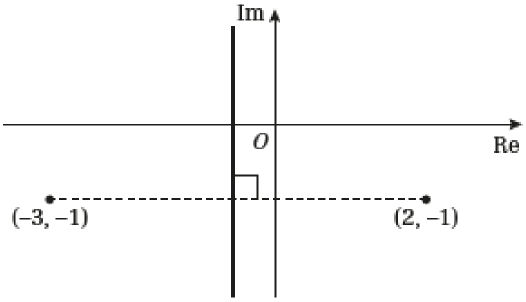
$d=\frac{3\sqrt{2}}{2}$

Therefore $z=-\frac{3\sqrt{2}}{2}+\left(3+\frac{3\sqrt{2}}{2}\right)\text{i}$

41 $\arg(z+3+\text{i})=\frac{\pi}{2}$ is the half-line originating at $(-3,-1)$ at $\frac{\pi}{2}$ to the positive x-axis



42 a $|z+3+\mathrm{i}|=|z-2+\mathrm{i}|$



b $|z|_{\min}=\dfrac{1}{2}$

c $\arg z=-\dfrac{3\pi}{4}$ is the half-line originating at (0, 0) at $-\dfrac{3\pi}{4}$ to the positive x-axis

$\arg z=-\dfrac{3\pi}{4}$ is part of the line $y=x$

Substituting $x=-\dfrac{1}{2}$ into $y=x$ gives $y=-\dfrac{1}{2}$

Therefore $z=-\dfrac{1}{2}-\dfrac{1}{2}\mathrm{i}$

43 a The locus forms a major arc since $\theta = \frac{\pi}{4} < \frac{\pi}{2}$

b Let $z = x + \mathrm{i}y$

Then $\frac{z+\mathrm{i}}{z-\mathrm{i}} = \frac{x+\mathrm{i}(y+1)}{x+\mathrm{i}(y-1)} = \frac{(x+\mathrm{i}(y+1))(x-\mathrm{i}(y-1))}{x^2+(y-1)^2} = \frac{x^2+y^2-1+2\mathrm{i}x}{x^2+(y-1)^2}$

$= \frac{x^2+y^2-1}{x^2+(y-1)^2} + \mathrm{i}\left(\frac{2x}{x^2+(y-1)^2}\right)$

Now $\arg\left(\frac{z+\mathrm{i}}{z-\mathrm{i}}\right) = \frac{\pi}{4}$, so $\tan\left(\arg\left(\frac{z+\mathrm{i}}{z-\mathrm{i}}\right)\right) = \tan\frac{\pi}{4} = 1$

Therefore $\dfrac{\frac{2x}{x^2+(y-1)^2}}{\frac{x^2+y^2-1}{x^2+(y-1)^2}} = 1$

$\frac{2x}{x^2+(y-1)^2} = \frac{x^2+y^2-1}{x^2+(y-1)^2}$

$x^2+y^2-1 = 2x$

$x^2-2x+y^2-1 = 0$

$(x-1)^2-1+y^2-1 = 0$

$(x-1)^2+y^2 = 2$

Hence the centre is at $(1, 0)$

44 a A is represented by the complex number $-1+3\mathrm{i}$.

b The radius of the circle is given by $\left|\overrightarrow{XA}\right| = \left|-1-2\mathrm{i}-(-1+3\mathrm{i})\right| = |-5\mathrm{i}| = 5$

So the area of the sector is $\frac{1}{2}r^2\theta = \frac{25\pi}{12}$

Substituting $r = 5$ gives $\frac{25\theta}{2} = \frac{25\pi}{12}$

Therefore $\theta = \frac{\pi}{6}$

Now $\overrightarrow{XB} = -5\sin\frac{\pi}{6} + 5\cos\frac{\pi}{6}\mathrm{i} = -\frac{5}{2} + \frac{5\sqrt{3}}{2}\mathrm{i}$

Therefore $b = \overrightarrow{OB} = -1-2\mathrm{i} - \frac{5}{2} + \frac{5\sqrt{3}}{2}\mathrm{i}$

$= -\frac{7}{2} + \frac{5\sqrt{3}-4}{2}\mathrm{i}$

45 Let $z = x + \mathrm{i}y$

$|z-1| = \sqrt{2}|z-\mathrm{i}|$

$|(x-1)+\mathrm{i}y| = \sqrt{2}|x+(y-1)\mathrm{i}|$

Squaring the modulus gives

$|x-1+\mathrm{i}y|^2 = 2|x+(y-1)\mathrm{i}|^2$

$(x-1)^2 + y^2 = 2\left(x^2 + (y-1)^2\right)$

$x^2 - 2x + 1 + y^2 = 2x^2 + 2y^2 - 4y + 2$

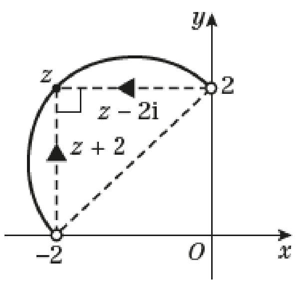
$x^2 + 2x + y^2 - 4y + 1 = 0$

$(x+1)^2 - 1 + (y-2)^2 - 4 + 1 = 0$

$(x+1)^2 + (y-2)^2 = 4$

Hence the circle has radius 2 and centre $(-1, 2)$

46 a

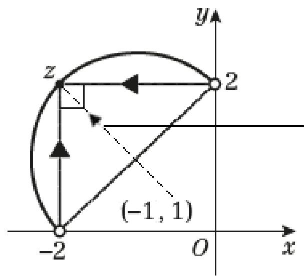


$\arg\left(\dfrac{z-2\mathrm{i}}{z+2}\right) = \arg(z-2\mathrm{i}) - \arg(z+2) = \dfrac{\pi}{2}$.

The angles which the vectors make with the positive x-axis differ by a right angle. As drawn here, the difference is $\pi - \dfrac{\pi}{2} = \dfrac{\pi}{2}$. The locus of the points, where the difference is a right angle, is a semi-circle, with the line joining –2 on the real axis to 2 on the imaginary axis as diameter.

It is a common error to complete the circle. The lower right hand completion of the circle has equation $\arg\left(\dfrac{z-2\mathrm{i}}{z+2}\right) = -\dfrac{\pi}{2}$.

b



The dotted line represents the complex number $z+1-\mathrm{i} = z-(-1+\mathrm{i})$. The length of this vector is the radius of the circle.

The diameter of the circle is given by $d^2 = 2^2 + 2^2 = 8$, so $d = 2\sqrt{2}$

Therefore $|z+1-\mathrm{i}| = \dfrac{2\sqrt{2}}{2} = \sqrt{2}$

47 a Both loci L and M are circles, hence they are similar.

47 b Computing the scale factor of enlargement amounts to computing the radii of both circles.

For L we have:

$$|z-4| = \sqrt{5}|z+2\text{i}|$$
$$|(x-4)+\text{i}y| = \sqrt{5}|x+(y+2)\text{i}|$$
$$|(x-4)+\text{i}y|^2 = 5|x+(y+2)\text{i}|^2$$
$$(x-4)^2 + y^2 = 5\left(x^2+(y+2)^2\right)$$
$$x^2 - 8x + 16 + y^2 = 5x^2 + 5y^2 + 20y + 20$$
$$4x^2 + 8x + 4y^2 + 20y + 4 = 0$$
$$x^2 + 2x + y^2 + 5y + 1 = 0$$
$$(x+1)^2 - 1 + \left(y+\tfrac{5}{2}\right)^2 - \tfrac{25}{4} + 1 = 0$$

Which simplifies to

$$(x+1)^2 + \left(y+\tfrac{5}{2}\right)^2 = \tfrac{25}{4}$$

Hence the radius of L is $\frac{5}{2}$

For M we have:

$$|z-6| = \sqrt{7}|z+6\text{i}|$$
$$|(x-6)+\text{i}y| = \sqrt{7}|x+\text{i}(y+6)|$$
$$|(x-6)+\text{i}y|^2 = 7|x+(y+6)\text{i}|^2$$
$$(x-6)^2 + y^2 = 7x^2 + 7(y+6)^2$$
$$x^2 - 12x + 36 + y^2 = 7x^2 + 7y^2 + 84y + 252$$
$$6x^2 + 12x + 6y^2 + 84y + 216 = 0$$
$$x^2 + 2x + y^2 + 14y + 36 = 0$$
$$(x+1)^2 - 1 + (y+7)^2 - 49 + 36 = 0$$
$$(x+1)^2 + (y+7)^2 = 14$$

Hence this circle has radius $\sqrt{14}$

So the scale factor of enlargement is $\dfrac{\sqrt{14}}{\frac{5}{2}} = \dfrac{2\sqrt{14}}{5}$

48 The locus is given by

$$\arg\left(\frac{z+1}{z}\right)=\frac{\pi}{4}$$

Substituting $z=x+\mathrm{i}y$

$$\frac{z+1}{z}=\frac{x+1+\mathrm{i}y}{x+\mathrm{i}y}=\frac{(x+1+\mathrm{i}y)(x-\mathrm{i}y)}{(x+\mathrm{i}y)(x-\mathrm{i}y)}=\frac{x^2+x+\mathrm{i}xy-\mathrm{i}xy-\mathrm{i}y+y^2}{x^2+y^2}$$

$$=\frac{x^2+x+y^2-\mathrm{i}y}{x^2+y^2}=\frac{x^2+x+y^2}{x^2+y^2}+\frac{-y}{x^2+y^2}\mathrm{i}$$

Now $\arg\left(\frac{z+1}{z}\right)=\frac{\pi}{4}$, so $\tan\left(\arg\left(\frac{z+1}{z}\right)\right)=\tan\frac{\pi}{4}=1$

So $\dfrac{\frac{-y}{x^2+y^2}}{\frac{x^2+x+y^2}{x^2+y^2}}=1$

$$-y=x^2+x+y^2$$
$$x^2+x+y^2+y=0$$
$$\left(x+\tfrac{1}{2}\right)^2-\tfrac{1}{4}+\left(y+\tfrac{1}{2}\right)^2-\tfrac{1}{4}=0$$
$$\left(x+\tfrac{1}{2}\right)^2+\left(y+\tfrac{1}{2}\right)^2=\tfrac{1}{2}$$

So the centre of the circle is $\left(-\tfrac{1}{2},-\tfrac{1}{2}\right)$ and the radius is $\dfrac{1}{\sqrt{2}}$

Therefore, this is the major arc of a circle.

The length of the curve required is $r(2\pi-\theta)$, where θ is the angle between the lines connecting the endpoints of the arc to the centre of the circle.

Geometrically we can see that $\tan\dfrac{\theta}{2}=\dfrac{\frac{1}{2}}{\frac{1}{2}}=1$, so $\theta=\dfrac{\pi}{2}$

Therefore the length of the arc is $\dfrac{1}{\sqrt{2}}\left(2\pi-\dfrac{\pi}{2}\right)=\dfrac{1}{\sqrt{2}}\left(\dfrac{3\pi}{2}\right)=\dfrac{3\pi}{2\sqrt{2}}$

49 We consider the locus $|z-\mathrm{i}| = \sqrt{p}\,|z+1|$

Squaring gives $|z-\mathrm{i}|^2 = p|z+1|^2$

Substituting $z = x + \mathrm{i}y$:

$$|x+(y-1)\mathrm{i}|^2 = p|(x+1)+\mathrm{i}y|^2$$

$$x^2+(y-1)^2 = p\left((x+1)^2+y^2\right)$$

$$x^2+y^2-2y+1 = px^2+2px+p+py^2$$

$$(p-1)x^2+2px+2y+(p-1)y^2+p-1=0$$

$$x^2+\frac{2px}{p-1}+y^2+\frac{2y}{p-1}+1=0$$

$$\left(x+\frac{p}{p-1}\right)^2-\frac{p^2}{(p-1)^2}+\left(y+\frac{1}{p-1}\right)^2-\frac{1}{(p-1)^2}+1=0$$

$$\left(x+\frac{p}{p-1}\right)^2+\left(y+\frac{1}{p-1}\right)^2=\frac{p^2}{(p-1)^2}+\frac{1}{(p-1)^2}-1$$

$$\left(x+\frac{p}{p-1}\right)^2+\left(y+\frac{1}{p-1}\right)^2=\frac{p^2+1-(p-1)^2}{(p-1)^2}$$

$$\left(x+\frac{p}{p-1}\right)^2+\left(y+\frac{1}{p-1}\right)^2=\frac{2p}{(p-1)^2}$$

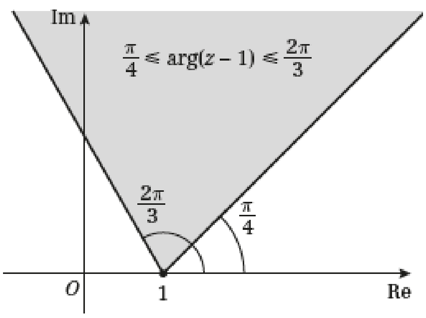
Hence the radius is $\dfrac{\sqrt{2p}}{p-1}$

For a circumference of 24π the radius is 12

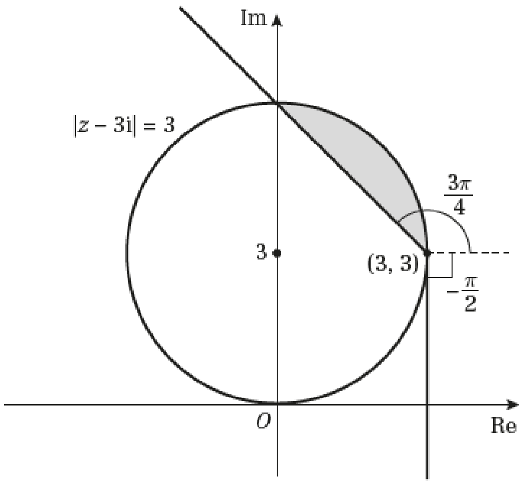
Hence $\dfrac{\sqrt{2p}}{p-1}=12$

$$2p=(12p-12)^2$$

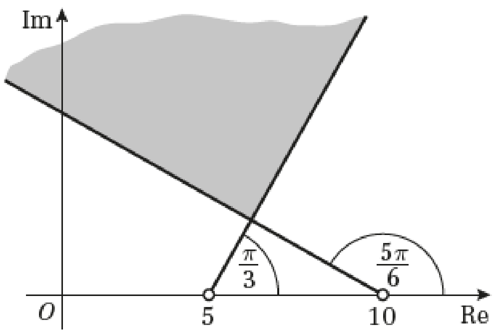
50 $\dfrac{\pi}{4} \le \arg(z-1) \le \dfrac{2\pi}{3}$



51 $\left\{z \in \mathbb{C} : -\frac{\pi}{2} < \arg(z - 3 - 3\mathrm{i}) \leq \frac{3\pi}{4}\right\} \cap \{z \in \mathbb{C} : |z - 3\mathrm{i}| \leq 3\}$



52



53 a The locus is given by the inequalities $|z - 6\mathrm{i}| \leqslant 2|z - 3|$ and $\mathrm{Re}(z) \leqslant k$

Taking the first inequality:

$|z - 6\mathrm{i}|^2 \leqslant 4|z - 3|^2$

$|x + (y - 6)\mathrm{i}|^2 \leqslant 4|x - 3 + \mathrm{i}y|^2$

$x^2 + (y - 6)^2 \leqslant 4\left((x - 3)^2 + y^2\right)$

$x^2 + y^2 - 12y + 36 \leqslant 4x^2 - 24x + 36 + 4y^2$

$0 \leqslant 3x^2 - 24x + 3y^2 + 12y$

$0 \leqslant x^2 - 8x + y^2 + 4y$

$x^2 - 8x + y^2 + 4y \geqslant 0$

$(x - 4)^2 - 16 + (y + 2)^2 - 4 \geqslant 0$

$(x - 4)^2 + (y + 2)^2 \geqslant 20$

Hence the circle has centre $(4, -2)$

Hence, for a semi-circle, we should take $k = 4$

b The area of the semicircle is $\frac{\pi r^2}{2} = \frac{\pi \times 20}{2} = 10\pi$

54 The line corresponding to $|z-p|=|z-q|$ is given by the perpendicular bisector of p and q,

that is $x=\dfrac{p+q}{2}$

The area of the triangular region is therefore $\dfrac{1}{2}\times\left(\dfrac{q-p}{2}\right)^2=x$

So $(q-p)^2=8x$

$q-p=\sqrt{8x}$

So $q=p+\sqrt{8x}$, as required

55 a The transformation defined by $w=3z+4-2\text{i}$ represents a scaling by 3, followed by a translation by the complex number $4-2\text{i}$.
The translation leaves the area of the triangle invariant.
Therefore the new area is $3^2\times 8=72$.

55 b We consider what happens to the line $\text{Im}(z)=4$ under the transformation.
Consider a point $z=x+4\text{i}$ on the line.
This is mapped to $w=3(x+4\text{i})+4-2\text{i}=3x+4+10\text{i}$.
Hence the line is mapped to the line $\text{Im}(z)=10$.

56 $w=\dfrac{2z-1}{z-2}\Rightarrow wz-2w=2z-1$

$wz-2z=2w-1\Rightarrow z(w-2)=2w-1$

$z=\dfrac{2w-1}{w-2}$

$|z|=1\Rightarrow\left|\dfrac{2w+1}{w-2}\right|=1$

You know that $|z|=1$ and you are trying to find out about w. So it is a good idea to change the subject of the formula to z. You can then put the modulus of the right hand side of the new formula, which contains w, equal to 1.

$|2w-1|=|w-2|$

Let $w=u+\text{i}v$

$|2(u+\text{i}v)-1|=|u+\text{i}v-2|$

$|(2u-1)+\text{i}2v|=|(u-2)+\text{i}v|$

It is not easy to interpret this locus geometrically and so it is sensible to transform the problem into algebra, using the rule that if $z=x+\text{i}y$, then $|z|^2=x^2+y^2$.

$|(2u-1)+\text{i}2v|^2=|(u-2)+\text{i}v|^2$

$(2u-1)^2+4v^2=(u-2)^2+v^2$

$4u^2-4u+1+4v^2=u^2-4u+4+v^2$

$3u^2+3v^2=3\Rightarrow u^2+v^2=1$

This is a circle centre O, radius 1 and has the equation
$|w|=1$ in the Argand plane.

Hence, the circle $|z|=1$ is mapped onto the circle
$|w|=1$, as required.

57 a $z = x + \frac{1}{2}\text{i}$

The real part of a complex number on Im $z = \frac{1}{2}$ can have any real value, which you can represent by the symbol x, but the imaginary part must be $\frac{1}{2}$.

$$w = \frac{z-\text{i}}{z}$$

$$zw = z - \text{i} \Rightarrow z - wz = \text{i}$$

$$z = \frac{\text{i}}{1-w}$$

Let $w = u + \text{i}v$

$$x + \frac{1}{2}\text{i} = \frac{\text{i}}{1-u-\text{i}v}$$

Multiplying the numerator and denominator by $1-u+\text{i}v$

Multiply the numerator and the denominator of the right hand side by the conjugate complex of $1-u-\text{i}v$ which is $1-u+\text{i}v$.

$$x + \frac{1}{2}\text{i} = \frac{\text{i}(1-u+\text{i}v)}{(1-u)^2+v^2},$$

$$= \frac{-v}{(1-u)^2+v^2} + \frac{1-u}{(1-u)^2+v^2}\text{i}$$

Equating imaginary parts

$$\frac{1}{2} = \frac{1-u}{u^2-2u+1+v^2}$$

You are aiming at $|w| = 1$. If $w = u + \text{i}v$, this is the equivalent to $u^2+v^2 = 1$. So that is the expression you are looking for.

$$u^2 - 2u + 1 + v^2 = 2 - 2u$$

$$u^2 + v^2 = 1$$

$u^2 + v^2 = 1$ is a circle centre O, radius 1.

Hence the line, Im $z = \frac{1}{2}$ is mapped onto the circle with equation $|w| = 1$.

b The transformation $w' = \frac{z-\text{i}}{z}$ maps the line Im $z = \frac{1}{2}$ onto the circle with centre O and radius 1.

The first transformation is the transformation in part **a**.

The transformation $w'' = 2w'$ maps the circle with centre O and radius 1 onto the circle with centre O and radius 2.

The transformation $z \mapsto kz$ increases the radius of the circle by a factor of k. This transformation is an enlargement, factor k, centre of enlargement O.

The transformation $w = w'' + 3 - \text{i}$ maps the circle with centre O and radius 2 onto the circle with centre $3 - \text{i}$ and radius 2.

The transformation $z \mapsto z + a$ maps a circle centre O to a circle centre a. This transformation is a translation.

Combining the transformations

$$w = 2\left(\frac{z-\text{i}}{z}\right) + 3 - \text{i}$$

$$= \frac{2z - 2\text{i} + 3z - \text{i}z}{z}$$

$$= \frac{(5-\text{i})z - 2\text{i}}{z}$$

58 a If $z = x + \mathrm{i}y$, then $\arg z = \frac{\pi}{4} \Rightarrow \frac{y}{x} = 1$

Let $x = y = \lambda$

$$w = \frac{\lambda + \lambda\mathrm{i} + 1}{\lambda + \lambda\mathrm{i} + \mathrm{i}} = \frac{(\lambda+1) + \lambda\mathrm{i}}{\lambda + (\lambda+1)\mathrm{i}}$$

$$|w| = \left|\frac{(\lambda+1)+\lambda\mathrm{i}}{\lambda+(\lambda+1)\mathrm{i}}\right| = \frac{|(\lambda+1)+\lambda\mathrm{i}|}{|\lambda+(\lambda+1)\mathrm{i}|}$$

For all complex numbers a and b, $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

$$= \frac{\left((\lambda+1)^2 + \lambda^2\right)^{\frac{1}{2}}}{\left(\lambda^2 + (\lambda+1)^2\right)^{\frac{1}{2}}} = 1$$

Hence the points on $\arg z = \frac{\pi}{4}$ map, under T, onto points on the circle $|w| = 1$.

As $\lambda > 0$, the image would only be part of this circle but the wording of the question does not require you to be more specific. You are only required to show that the image points are points on the circle; not all of the points on the circle. (The image is, in fact, just the lower right quadrant of the circle.)

b $wz + w\mathrm{i} = z + 1$

$wz - z = 1 - \mathrm{i}w$

$$z = \frac{1 - \mathrm{i}w}{w - 1}$$

$$|z| = \frac{|1 - \mathrm{i}w|}{|w - 1|} = 1$$

Hence $|1 - \mathrm{i}w| = |w - 1|$

This is the image under T of $|z| = 1$ but it is difficult to interpret and part **c** would be difficult without some further working.

$|1 - \mathrm{i}w| = |-\mathrm{i}(w + 1)| = |-\mathrm{i}||w + \mathrm{i}| = 1 \times |w + \mathrm{i}| = |w + \mathrm{i}|$

The image of $|z| = 1$ in the z-plane is

$|w + \mathrm{i}| = |w - 1|$

in the w-plane.

This is the locus of points equidistant from the points in the Argand plane representing $-\mathrm{i}$ and one. That is the perpendicular bisector of $(0, -1)$ and $(1, 0)$.

Writing $w = u + \mathrm{i}v$:

$|u + \mathrm{i}v + \mathrm{i}| = |u + \mathrm{i}v - 1|$

$|u + (v+1)\mathrm{i}| = |(u-1) + \mathrm{i}v|$

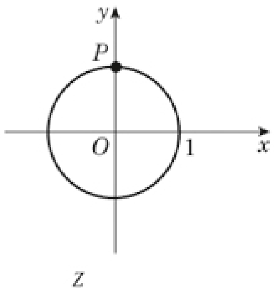
$|u + (v+1)\mathrm{i}|^2 = |(u-1) + \mathrm{i}v|^2$

$u^2 + (v+1)^2 = (u-1)^2 + v^2$

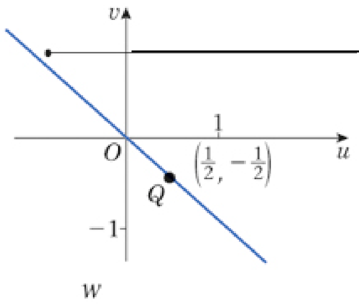
$u^2 + v^2 + 2v + 1 = u^2 - 2u + 1 + v^2$

So $v = -u$

58 c, d



$$z = \mathrm{i} \Rightarrow w = \frac{1+\mathrm{i}}{2\mathrm{i}} = \frac{\mathrm{i}+1}{2\mathrm{i}} = \frac{1}{2} - \frac{1}{2}\mathrm{i}$$



The perpendicular bisector of $(0,-1)$ and $(1,0)$ is the line $v = -u$.

59 a $z = a^{-1}\mathrm{e}^{\mathrm{i}\theta}$

b Let $z = a^{-1}\mathrm{e}^{\mathrm{i}\theta}$ then we have $w = az + \dfrac{1}{z} = \mathrm{e}^{\mathrm{i}\theta} + a\mathrm{e}^{-\mathrm{i}\theta}$

$= \cos\theta + \mathrm{i}\sin\theta + a(\cos\theta - \mathrm{i}\sin\theta)$

$= (1+a)\cos\theta + (1-a)\mathrm{i}\sin\theta$

$= u + \mathrm{i}v$

Hence $u = (1+a)\cos\theta$ and $v = (1-a)\sin\theta$

$$\left(\frac{u}{1+a}\right)^2 + \left(\frac{v}{1-a}\right)^2 = 1$$

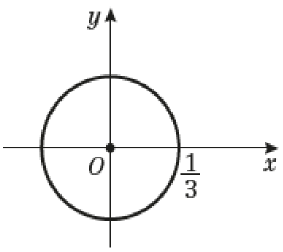
$$u^2(1-a)^2 + v^2(1+a)^2 = (1+a)^2(1-a)^2$$

$$u^2(1-a)^2 + v^2(1+a)^2 = \left[(1+a)(1-a)\right]^2$$

$$u^2(1-a)^2 + v^2(1+a)^2 = (1-a^2)^2$$

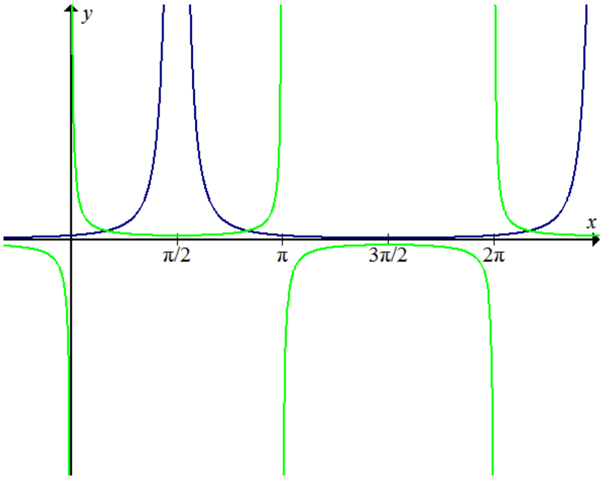
as required

c This ellipse corresponds to the case $a = 3$, hence the points on the z-plane that are transformed to the ellipse are those such that $|z| = \frac{1}{3}$



Challenge

1 $\frac{1}{1-\sin x} < \frac{1}{\sin x}$ for $0 < x < 2\pi$



Equating $y = \frac{1}{1-\sin x}$ and $y = \frac{1}{\sin x}$ gives:

$\sin x = 1 - \sin x$

$2\sin x = 1$

$\sin x = \frac{1}{2}$

In the range $0 < x < 2\pi$

$\sin x = \frac{1}{2}$ at $\frac{\pi}{6}$ and $\frac{5\pi}{6}$

Therefore:

$0 < x < \frac{\pi}{6}$ and $\frac{5\pi}{6} < x < \pi$

2 a $\omega = e^{\frac{2\pi i}{3}}, \omega^{3k} = e^{2\pi i k} = 1$

$$n = 0: \frac{1^n + \omega^n + (\omega^2)^n}{3} = \frac{1+1+1}{3} = 1$$

$$n = 3k: \frac{1^{3k} + \omega^{3k} + (\omega^2)^{3k}}{3} = \frac{1^{3k} + \omega^{3k} + (\omega^{3k})^2}{3} = \frac{1+1+1}{3} = 1$$

$$n = 3k+1: \frac{1^{3k+1} + \omega^{3k+1} + (\omega^2)^{3k+1}}{3} = \frac{1 + \omega^{3k+1} + \omega^{6k+2}}{3} = \frac{1 + \omega^{3k}\omega + (\omega^{3k})^2\omega^2}{3} = \frac{1 + \omega + \omega^2}{3} = 0$$

$$n = 3k+2: \frac{1^{3k+1} + \omega^{3k+2} + (\omega^2)^{3k+2}}{3} = \frac{1 + \omega^{3k+2} + \omega^{6k+4}}{3} = \frac{1 + \omega^{3k}\omega^2 + (\omega^{3k})^2\omega^3\omega}{3} = \frac{1 + \omega^2 + \omega}{3} = 0$$

b $\mathrm{f}(x)=\sum a_n x^n$

$\mathrm{f}(1)=\sum a_n, \mathrm{f}(\omega)=\sum a_n \omega^n, \mathrm{f}(\omega^2)=\sum a_n \left(\omega^2\right)^n$

$$\frac{\mathrm{f}(1)+\mathrm{f}(\omega)+\mathrm{f}(\omega^2)}{3}=\sum a_n \frac{1^n+\omega^n+\left(\omega^2\right)^n}{3}$$

$$=\sum a_n 1_{\{n=3k\}},$$

where $1_{\{n=3k\}}=1$ if $n=3k$ and 0 otherwise

c $\mathrm{f}(x)=(1+x)^{45}=\sum_{r=0}^{45}\binom{45}{r}x^r$

$$S=\sum_{r=0}^{45}\binom{45}{r}1_{\{r=3k\}}=\sum_{r=0}^{15}\binom{45}{3r}$$

$$=\frac{\mathrm{f}(1)+\mathrm{f}(\omega)+\mathrm{f}(\omega^2)}{3}=\frac{2^{45}+(1+\omega)^{45}+(1+\omega^2)^{45}}{3}$$

$$=\frac{2^{45}+(-\omega^2)^{45}+(-\omega)^{45}}{3}=\frac{2^{45}-(\omega^3)^{30}-(\omega^3)^{15}}{3}$$

$$=\frac{2^{45}-2}{3} \text{ as } 1+\omega+\omega^2=0 \text{ and } \omega^3=0$$

Review exercise 2

1 The integrating factor is

$$e^{\int \frac{4}{x} dx} = e^{4\ln x} = e^{\ln x^4} = x^4$$

If the differential equation has the form $\frac{dy}{dx} + Py = Q$, the integrating factor is $e^{\int P dx}$.

For any function $f(x)$, $e^{\ln f(x)} = f(x)$.

Multiply the equation throughout by x^4

$$x^4 \frac{dy}{dx} + 4x^3 y = 6x^5 - 5x^4$$

$$\frac{d}{dx}(x^4 y) = 6x^5 - 5x^4$$

$$x^4 y = \int (6x^5 - 5x^4) dx = x^6 - x^5 + C$$

$$y = x^2 - x + \frac{C}{x^4}$$

It is important that you remember to add the constant of integration. When you divide by x^4, the constant becomes a function of x and its omission would be a significant error.

2 The integrating factor is

$$e^{\int -\frac{1}{x} dx} = e^{-\ln x} = e^{\ln \frac{1}{x}} = \frac{1}{x}$$

For all n, $n\ln x = \ln x^n$, so for $n = -1$, $-\ln x = \ln x^{-1} = \ln \frac{1}{x}$.

Multiply the equation throughout by $\frac{1}{x}$

$$\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = x$$

$$\frac{d}{dx}\left(\frac{y}{x}\right) = x$$

The product rule for differentiating, in this case $\frac{d}{dx}\left(y \times \frac{1}{x}\right) = \frac{dy}{dx} \times \frac{1}{x} + y \times -\frac{1}{x^2}$, enables you to write the differential equation as an exact equation, where one side is the exact derivative of a product and the other side can be integrated with respect to x.

$$\frac{y}{x} = \frac{x^2}{2} + C$$

$$y = \frac{x^3}{2} + Cx$$

3 $(x+1)\dfrac{dy}{dx}+2y=\dfrac{1}{x}$

$$\frac{dy}{dx}+\frac{2}{x+1}y=\frac{1}{x(x+1)}$$

If the equation is in the form $R\dfrac{dy}{dx}+Sy=T$, you must begin by dividing throughout by R, in this case $(x+1)$, before finding the integrating factor.

The integrating factor is

$$e^{\int\frac{2}{x+1}dx}=e^{2\ln(x+1)}=e^{\ln(x+1)^2}=(x+1)^2$$

Multiply throughout by $(x+1)^2$

$$(x+1)^2\frac{dy}{dx}+2(x+1)y=\frac{x+1}{x}$$

To integrate $\dfrac{x+1}{x}$, write $\dfrac{x+1}{x}=\dfrac{x}{x}+\dfrac{1}{x}=1+\dfrac{1}{x}$.

$$\frac{d}{dx}\left((x+1)^2y\right)=1+\frac{1}{x}$$

$$(x+1)^2y=\int\left(1+\frac{1}{x}\right)dx=x+\ln x+C$$

$$y=\frac{x+\ln x+C}{(x+1)^2}$$

You divide throughout by $(x+1)^2$ to obtain the equation in the form $y=f(x)$. This is required by the wording of the question.

4 The integrating factor is $e^{\int\tan x\,dx}$

$$\int\tan x\,dx=\int\frac{\sin x}{\cos x}dx=-\ln\cos x=\ln\frac{1}{\cos x}=\ln\sec x$$

Here we have that $\int\dfrac{f'(x)}{f(x)}dx=\ln f(x)$. As $-\sin x$ is the derivative of $\cos x$, $\int\dfrac{-\sin x}{\cos x}dx=\ln\cos x$.

Hence

$$e^{\int\tan x\,dx}=e^{\ln\sec x}=\sec x$$

Multiply the differential equation throughout by $\sec x$

$$\sec x\frac{dy}{dx}+y\sec x\tan x=e^{2x}\sec x\cos x=e^{2x}$$

$\sec x\cos x=\dfrac{1}{\cos x}\times\cos x=1$

$$\frac{d}{dx}(y\sec x)=e^{2x}$$

$$y\sec x=\int e^{2x}dx=\frac{e^{2x}}{2}+C$$

Multiply throughout by $\cos x$

$$y=\left(\frac{e^{2x}}{2}+C\right)\cos x$$

$y=2$ at $x=0$

$$2=\frac{1}{2}+C\Rightarrow C=\frac{3}{2}$$

The condition $y=2$ at $x=0$ enables you to evaluate the constant of integration and find the particular solution of the differential equation for these values.

$$y=\frac{1}{2}\left(e^{2x}+3\right)\cos x$$

5 The integrating factor is $e^{\int 2\cot 2x\,dx}$

$$\int 2\cot 2x\,dx = \int \frac{2\cos 2x}{\sin 2x}\,dx = \ln \sin 2x$$

Hence

$$e^{\int 2\cot 2x\,dx} = e^{\ln \sin 2x} = \sin 2x$$

Multiply the differential equation throughout by $\sin 2x$

$$\sin 2x \frac{dy}{dx} + 2y\cos 2x = \sin x \sin 2x$$

Using the identity $\sin 2x = 2\sin x\cos x$.

$$\frac{d}{dx}(y\sin 2x) = 2\sin^2 x\cos x$$

$$y\sin 2x = \frac{2\sin^3 x}{2} + C$$

$$y = \frac{2\sin^3 x}{3\sin 2x} + \frac{C}{\sin 2x}$$

As $\frac{d}{dx}(\sin^3 x) = 3\sin^2 x\cos x$, then $\int \sin^2 x\cos dx = \frac{\sin^3 x}{3}$. It saves time to find integrals of this type by inspection. However, you can use the substitution $\sin^y = s$ if you find inspection difficult.

6 $(1+x)\dfrac{dy}{dx} - xy = xe^{-x}$

$\dfrac{dy}{dx} - \dfrac{xy}{1+x} = \dfrac{xe^{-x}}{1+x}$ **(1)**

If the equation is in the form $R\dfrac{dy}{dx} + Sy = T$, you must begin by dividing throughout by R, in this case $(1+x)$ before finding the integrating factor.

The integrating factor is $e^{\int -\frac{x}{1+x}dx}$

$\dfrac{x}{1+x} = \dfrac{1+x-1}{1+x} = 1 - \dfrac{1}{1+x}$

To integrate an expression in which the degree of the numerator is greater or equal to the degree of the denominator, you must transform the expression into one with a proper fraction. This can be done using partial fractions, long division or, as here, using decomposition.

Hence

$\displaystyle\int \frac{x}{1+x}dx = x - \ln(1+x)$

and the integrating factor is

$e^{-x+\ln(1+x)} = e^{-x}e^{\ln(1+x)} = e^{-x} + (1+x)$

Multiplying (**1**) throughout by $(1+x)e^{-x}$

$(1+x)e^{-x}\dfrac{dy}{dx} - xe^{-x}y = xe^{-2x}$

$\dfrac{d}{dx}\left(y(1+x)e^{-x}\right) = xe^{-2x}$

$y(1+x)e^{-x} = \int xe^{-2x}dx$

You integrate $x\,e^{-2x}$ using integration by parts.

$= -\dfrac{xe^{-2x}}{2} + \displaystyle\int \frac{e^{-2x}}{2}dx = -\frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4} + C$

$\dfrac{e^{-2x}}{e^{-x}} = e^{-2x-(-x)} = e^{-2x+x} = e^{-x}$

$y = -\dfrac{xe^{-2x}}{2(1+x)e^{-x}} - \dfrac{e^{-2x}}{4(1+x)e^{-x}} + \dfrac{C}{(1+x)e^{-x}}$

$= -\dfrac{xe^{-x}}{2(1+x)} - \dfrac{e^{-x}}{4(1+x)} + \dfrac{Ce^{x}}{(1+x)}$

$y = 1$ at $x = 0$

$1 = 0 - \dfrac{1}{4} + C \Rightarrow C = \dfrac{5}{4}$

$y = \dfrac{5e^{x}}{4(1+x)} - \dfrac{xe^{-x}}{2(1+x)} - \dfrac{e^{-x}}{4(1+x)}$

This expression could be put over a common denominator but, other than requiring that y is expressed in terms of x, the question asks for no particular form and this is an acceptable answer.

7 a Dividing throughout by $\cos x$

$$\frac{dy}{dx} + \frac{\sin x}{\cos x} y = \cos^2 x \qquad \textbf{(1)}$$

$$\int \frac{\sin x}{\cos x} dx = -\int \frac{-\sin x}{\cos x} dx = -\ln \cos x = \ln \frac{1}{\cos x} = \ln \sec x$$

Here we use $\int \frac{f'(x)}{f(x)} dx = \ln f(x)$. As $-\sin x$ is the derivative if $\cos x$, $-\int \frac{-\sin x}{\cos x} dx = -\ln \cos x$.

As $\ln 1 = 0$,

$-\ln \cos x = \ln 1 - \ln \cos x = \ln \frac{1}{\cos x}$,

using the log law $\ln a - \ln b = \ln \frac{a}{b}$.

Hence the integrating factor is $e^{\ln \sec x} = \sec x$

Multiply **(1)** by $\sec x$

$$\sec x \frac{dy}{dx} + \sec x \frac{\sin x}{\cos x} y = \cos^2 x \sec x$$

$$\sec x \frac{dy}{dx} + (\sec x \tan x) y = \cos x$$

$$\frac{d}{dx}(y \sec x) = \cos x$$

$$y \sec x = \int \cos x \, dx = \sin x + C$$

Multiplying throughout by $\cos x$

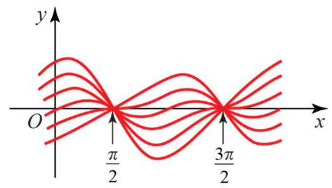
$$y = \sin x \cos x + C \cos x$$

b Where $\cos x = 0$ and $0 \leqslant x \leqslant 2\pi$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

The points $\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{3\pi}{2}, 0\right)$ lie on all of the solution curves of the differential equation.

In general, for a given value of x, different values of c give different values of y. However, if $\cos x = 0$, the c will have no effect and y will be zero for any value of c.



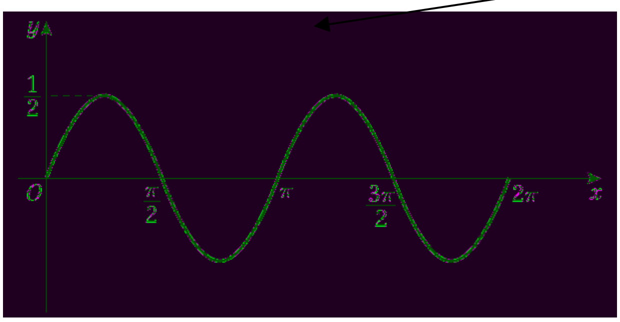
c $y = \sin x \cos x + \cos x$

At $x = 0, y = 0$

$0 = 0 + C \Rightarrow C = 0$

$y = \sin x \cos x = \frac{1}{2} \sin 2x$

Using the identity $\sin 2x = 2 \sin x \cos x$. $\sin 2x$ is a function with period π. So the curve makes two complete oscillations in the interval $0 \leqslant x \leqslant 2\pi$



8 **a** The integrating factor is

$$e^{\int 2\,dx} = e^{2x}$$

Multiplying the differential equation throughout by e^{2x}

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = xe^{2x}$$

$$\frac{d}{dx}(ye^{2x}) = xe^{2x}$$

$$ye^{2x} = \int xe^{2x}\,dx$$

$$= \frac{xe^{2x}}{2} - \int \frac{e^{2x}}{2}\,dx = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C$$

$$y = \frac{x}{2} - \frac{1}{4} + Ce^{-2x}$$

Integrate by parts.

b $y = 1$ at $x = 0$

$$1 = 0 - \tfrac{1}{4} + C \Rightarrow C = \tfrac{5}{4}$$

$$y = \frac{x}{2} - \frac{1}{4} + \frac{5e^{-2x}}{4}$$

This is the particular solution of the differential equation for $y = 1$ at $x = 0$. You are asked to sketch this in part **c**.

For a minimum $\dfrac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{1}{2} - \frac{5e^{-2x}}{2} = 0 \Rightarrow 5e^{-2x} = 1 \Rightarrow e^{2x} = 5$$

$$\ln e^{2x} = \ln 5 \Rightarrow 2x = \ln 5$$

$$x = \tfrac{1}{2}\ln 5$$

At the minimum, the differential equation reduces to

$$2y = x$$

The differential equation is $\dfrac{dy}{dx} + 2y = x$. At the minimum, $\dfrac{dy}{dx} = 0$ and so $2y = x$. If you did not see this you could, of course, substitute $x = \dfrac{1}{2}\ln 5$ into the particular solution and find y. This would take longer but would gain full marks.

Hence

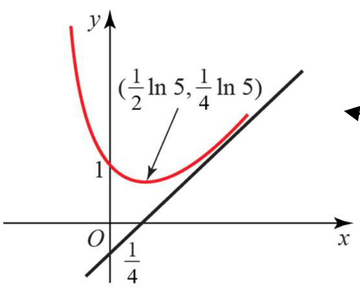
$$y = \tfrac{1}{2}x = \tfrac{1}{4}\ln 5$$

$$\frac{d^2y}{d^2x} = 5e^{-2x} > 0 \text{ for any real } x$$

This confirms the point is a minimum.

The coordinates of the minimum are $\left(\tfrac{1}{2}\ln 5, \tfrac{1}{4}\ln 5\right)$.

c



As x increases, $e^{-2x} \to 0$ and so $\dfrac{x}{2} - \dfrac{1}{4} + \dfrac{5e^{-2x}}{4} \to \dfrac{x}{2} - \dfrac{1}{4}$. This means that $y = \dfrac{x}{2} - \dfrac{1}{4}$ is an asymptote of the curve.

This has been drawn on the graph. It is not essential to do this, but if you recognise that this line is an asymptote, it helps you to draw the correct shape of the curve.

9 The auxiliary equation is

$$m^2+4m+5=0$$
$$m^2+4m+4=-1$$
$$(m+2)^2=-1$$
$$m=-2\pm\mathrm{i}$$

The general solution is

$$\theta=\mathrm{e}^{-2t}(A\cos t+B\sin t)$$

If the solutions to the auxiliary equation are $\alpha\pm\mathrm{i}\beta$, you may quote the result that the general solution of the differential equation is $\mathrm{e}^{\alpha t}.(A\cos\beta t+B\sin\beta t)$.

$$t=0,\theta=3$$
$$3=A$$

Using $\sin 0=0$ and $\cos 0=1$.

$$\frac{\mathrm{d}\theta}{\mathrm{d}t}=-2\mathrm{e}^{-2t}(A\cos t+B\sin t)+\mathrm{e}^{-2t}(-A\sin t+B\cos t)$$
$$t=0,\frac{\mathrm{d}\theta}{\mathrm{d}t}=-6$$
$$-6=-2A+B$$
$$B=2A-6=0$$

As $A=3$

The particular solution is

$$\theta=3\mathrm{e}^{-2t}\cos t$$

10 a $y=3x\sin 2x\Rightarrow\dfrac{\mathrm{d}y}{\mathrm{d}x}=3\sin 2x+6x\cos 2x$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=6\cos 2x+6\cos 2x-12x\sin 2x$$
$$=12\cos 2x-12x\sin 2x$$

Use the product rule for differentiating.

Substituting into the differential equation

$$12\cos 2x-\cancel{12x\sin 2x}+\cancel{12x\sin 2x}=k\cos 2x$$

Hence

$$k=12$$

10 b The auxiliary equation is

$m^2 + 4 = 0$

$m = \pm 2\text{i}$

The complementary function is given by

$y = A\cos 2x + B\sin 2x$

If the solutions to the auxiliary equation are $m = \pm \alpha\text{i}$, you may quote the result that the complementary function is $A \cos \alpha x + B\sin \alpha x$.

From **a**, the general solution is

$y = A\cos 2x + B\sin 2x + 3x\sin 2x$

Part **a** of the question gives you that $3x \sin 2x$ is a particular integral of the differential equation and general solution = complementary function + particular integral.

$x = 0, y = 2$

$2 = A$

$x = \dfrac{\pi}{4}, y = \dfrac{\pi}{2}$

$\dfrac{\pi}{2} = A\cos\dfrac{\pi}{2} + B\sin\dfrac{\pi}{2} + 3\times\dfrac{\pi}{4}\sin\dfrac{\pi}{2}$

$\dfrac{\pi}{2} = B + \dfrac{3\pi}{4} \Rightarrow B = -\dfrac{\pi}{4}$

Use $\cos\dfrac{\pi}{2} = 0$ and $\sin\dfrac{\pi}{2} = 1$.

The particular solution is

$y = 2\cos 2x - \dfrac{\pi}{4}\sin 2x + 3x\sin 2x$

11 a $y = a + bx \Rightarrow \dfrac{dy}{dx} = b$ and $\dfrac{d^2y}{dx^2} = 0$

Substituting into the differential equation

$0 - 4b + 4a + 4bx = 16 + 4x$

Equating the coefficients of x

$4b - 4 \Rightarrow b = 1$

Equating the constant coefficients

$-4b + 4a = 16$

$-4 + 4a = 16 \Rightarrow a = 5$

Use $b = 1$.

$a = 5, b = 1$

11 b The auxiliary equation is

$$m^2 - 4m + 4 = 0$$
$$(m-2)^2 = 0$$
$$m = 2, \text{repeated}$$

The complementary function is given by

$$y = e^{2x}(A + Bx)$$

If the auxiliary equation has a repeated root α, then the complementary function is $e^{\alpha x}(A + Bx)$. You can quote this result.

The general solution is

$$y = e^{2x}(A + Bx) + 5 + x$$

general solution = complementary function + particular integral.

$$y = 8,\ x = 0$$
$$8 = A + 5 \Rightarrow A = 3$$
$$\frac{dy}{dx} = 2e^{2x}(A + Bx) + Be^{2x} + 1$$
$$\frac{dy}{dx} = 9, x = 0$$
$$9 = 2A + B + 1 \Rightarrow B = 8 - 2A = 2$$

Use $A = 3$

The Particular solution is

$$y = e^{2x}(3 + 2x) + 5 + x$$

12 a The auxiliary equation is

$$m^2+4m+5=0$$
$$m^2+4m+4=-1$$
$$(m+2)^2=-1$$
$$m=-2\pm\mathrm{i}$$

The complementary function is given by
$y=\mathrm{e}^{-2x}(A\cos x+B\sin x)$

For a particular integral, let $y=p\cos 2x+q\sin 2x$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-2p\sin 2x+2q\cos 2x$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-4p\cos 2x-4q\sin 2x$$

If the right hand side of the second order differential equation is a sine or cosine function, then you should try a particular integral of the form $p\cos\omega x+q\sin\omega x$, with an appropriate ω. Here $\omega=2$.

Substituting into the differential equation

$$-4p\cos 2x-4q\sin 2x-8p\sin 2x+8q\cos 2x+5p\cos 2x+5q\sin 2x=65\sin 2x$$
$$(-4p+8q+5p)\cos 2x+(-4q-8p+5q)\sin 2x=65\sin 2x$$

Equating the coefficients of cos $2x$ and sin $2x$

$\cos 2x$: $\quad -4p+8q+5p=0\Rightarrow p+8q=0 \quad$ **(1)**

$\sin 2x$: $\quad -4q-8p+5q=65\Rightarrow -8p+q=65 \quad$ **(2)**

$8p+64q=0 \quad$ **(3)**

$65q=65\Rightarrow q=1$

The coefficients of cos$2x$ and sin $2x$ can be equated separately. The coefficient of cos$2x$ on the right hand side of this equation is zero.

Multiply **(1)** by 8 and add the result to **(2)**.

Substitute $q=1$ into **(1)**

$$p+8=0\Rightarrow p=-8$$

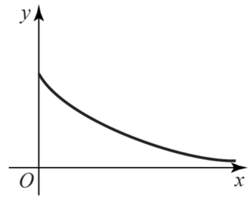
A particular integral is $-8\cos 2x+\sin 2x$

The general solution is

$$y=\mathrm{e}^{-2x}(A\cos x+B\sin x)+\sin 2x-8\cos 2x$$

12 b As $x \to \infty, e^{-2x} \to 0$ and, hence,

$y \to \sin 2x - 8\cos 2x$



The graph of e^{-2x} against x has this shape. As x becomes larger e^{-2x} is close to zero, so e^{-2x} (A cos x + B sin x) is also small.

Let

$$\sin 2x - 8\cos 2x = R\sin(2x - \alpha)$$
$$= R\sin 2x\cos\alpha - R\cos 2x\sin\alpha$$

Equating the coefficients of cos $2x$ and sin $2x$

$1 = R\cos\alpha \ldots$ **(4)**

$8 = R\sin\alpha \ldots$ **(5)**

$R^2\cos^2\alpha + R^2\sin^2\alpha = 1^2 + 8^2 = 65$

$R^2 = 65 \Rightarrow R\sqrt{65}$

Add **(4)** squared to **(5)** squared and use the identity $\cos^2\alpha + \sin^2\alpha = 1$.

$\dfrac{R\sin\alpha}{R\cos\alpha} = \dfrac{8}{1} \Rightarrow \tan\alpha = 8$

Divide **(5)** by **(4)**.

Hence, for large x, y can be approximated by the sine function $\sqrt{65}$ sin $(2x - a)$, where tan $\alpha = 8$ $(a \approx 82.9^\circ)$

13 a The auxiliary equation is

$$m^2 + 2m + 2 = 0$$
$$m^2 + 2m + 1 = -1$$
$$(m+1)^2 = -1$$
$$m = -1 \pm \mathrm{i}$$

The complementary function is

$y = e^{-t}(A\cos t + B\sin t)$

Try a particular integral $y = k\,e^{-t}$

If the right hand side of the differential equation is λe^{at+b}, where λ is any constant, then a possible form of the particular integral is $k\,e^{at+b}$.

$\dfrac{dy}{dt} = -k\,e^{-t}, \quad \dfrac{d^2y}{dt^2} = k\,e^{-t}$

Substituting into the differential equation

$k\,e^{-t} - 2k\,e^{-t} + 2k\,e^{-t} = 2e^{-t}$

Divide throughout by e^{-t}.

$k - 2k + 2k = 2 \Rightarrow k = 2$

A particular integral is $2e^{-t}$

The general solution is

$y = e^{-t}(A\cos t + B\sin t) + 2e^{-t}$

13 b $y=1, \quad t=0$

$1=A+2 \Rightarrow A=-1$

Substitute the boundary condition $y=1$, $t=0$ into the general solution gives you an equation for one arbitrary constant.

$\frac{dy}{dt}=-e^{-t}(A\cos t+B\sin t)+e^{-t}(-A\sin t+B\cos t)-2e^{-t}$

Use the product rule for differentiating.

$\frac{dy}{dt}=1, \ t=0$

$1=-A+B-2 \Rightarrow B=3+A=2$

As $A=-1$.

The particular solution is

$y=e^{-t}(2\sin t-\cos t)+2e^{-t}$

14 a The auxiliary equation is

$m^2+2m+5=0$

You may use any appropriate method to solve the quadratic. Completing the square works efficiently when the coefficient of m is given.

$m^2+2m+1=-4$

$(m+1)^2=-4$

$m=-1\pm 2i$

The general solution is

$x=e^{-t}(A\cos 2t+B\sin 2t)$

b $x=1, \ t=0$

$1=A$

Use the product rule for differentiation.

$\frac{dx}{dt}=-e^{-t}(A\cos 2t+B\sin 2t)+2e^{-t}(-A\sin 2t+B\cos 2t)$

$\frac{dx}{dt}=1, \ t=0$

$1=-A+2B \Rightarrow 2B=A+1=2 \Rightarrow B=1$

The particular solution is

$x=e^{-t}(\cos 2t+\sin 2t)$

Both A and B are 1.

14 c The curve crosses the t-axis where

$$e^{-t}(\cos 2t + \sin 2t) = 0$$

e^{-t} can never be zero.

$$\cos 2t + \sin 2t = 0$$

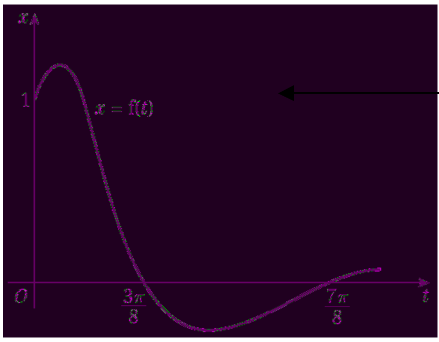
$$\sin 2t = -\cos 2t$$

$$\tan 2t = -1$$

Divide both sides by $\cos 2t$ and use the identity $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

$$2t = \frac{3\pi}{4}, \frac{7\pi}{4}$$

$$t = \frac{3\pi}{8}, \frac{7\pi}{8}$$



The boundary conditions give you that at $t = 0$, $x = 1$ and the curve has a positive gradient. The curve must then turn down and cross the axis at the two points where $t = \frac{3\pi}{8}$ and $\frac{7\pi}{8}$.

15 a The auxiliary equation is

$$2m^2 + 7m + 3 = 0$$

$$(2m+1)(m+3) = 0$$

$$m = -\tfrac{1}{2}, -3$$

The complementary function is given by

$$y = Ae^{-\frac{1}{2}t} + Be^{-3t}$$

If the auxiliary equation has two real solutions α and β, the complementary function is $y = Ae^{\alpha t} + Be^{\beta t}$. You can quote this result.

For a particular integral, try $y = at^2 + bt + c$

If the right hand side of the differential equation is a polynomial of degree n, then you can try a particular integral of the same degree. Here the right-hand side is a quadratic, so you try the general quadratic $at^2 + bt + c$.

$$\frac{dy}{dt} = 2at + b, \quad \frac{d^2y}{dt^2} = 2a$$

Substitute into the differential equation

$$4a + 14at + 7b + 3at^2 + 3bt + 3c = 3t^2 + 11t$$

$$3at^2 + (14a + 3b)t + 4a + 7b + 3c = 3t^2 + 11t$$

Equating the coefficients of t^2

$$3a = 3 \Rightarrow a = 1$$

Equating the coefficients of t

Use $a = 1$.

$$14a + 3b = 11 \Rightarrow 3b = 11 - 14a = -3 \Rightarrow b = -1$$

Equating the constant coefficients

Use $a = 1$ and $b = -1$.

$$4a + 7b + 3c = 0 \Rightarrow 3c = -4a - 7b = 3 \Rightarrow c = 1$$

A particular integral is $t^2 - t + 1$.

The general solution is $y = Ae^{-\frac{1}{2}t} + Be^{-3t} + t^2 - t + 1$.

15 b $y = 1,\ t = 0$

$1 = A + B + 1 \Rightarrow A + B = 0 \quad$ **(1)**

$\frac{dy}{dt} = -\frac{1}{2}Ae^{-\frac{1}{2}t} - 3Be^{-3t} + 2t - 1$

Differentiate the general solution in part **a** with respect to t.

$\frac{dy}{dt} = 1,\ t = 0$

$1 = -\frac{1}{2}A - 3B - 1 \Rightarrow \frac{1}{2}A + 3B = -2 \quad$ **(2)**

$A + 6B = -4 \quad$ **(3)**

$5B = -4 \Rightarrow B = -\frac{4}{5}$

Multiply (**2**) by 2 and then subtract (**1**) from (**3**).

Substituting $B = \frac{4}{5}$ into **(1)**

$A - \frac{4}{5} = 0 \Rightarrow A = \frac{4}{5}$

The particular solution is $y = \frac{4}{5}\left(e^{-\frac{1}{2}t} - e^{-3t}\right) + t^2 - t + 1$.

c When $t = 1,\ y = \frac{4}{5}\left(e^{-\frac{1}{2}} - e^{-3}\right) + 1 = 1.45\ (3\text{s.f.})$

16 a Let $y = \lambda x\cos 3x$

$\frac{dy}{dx} = \lambda\cos 3x - 3\lambda x\sin 3x$

$\frac{d^2y}{dx^2} = -3\lambda\sin 3x - 3\lambda\sin 3x - 9\lambda x\cos 3x$

$= -6\lambda\sin 3x - 9\lambda x\cos 3x$

Use the product rule for differentiation

$\frac{d}{dx}(x\sin 3x) = \frac{d}{dx}(x)\sin 3x + x\frac{d}{dx}(\sin 3x)$

$= \sin 3x + 3x\cos 3x$

Substituting into the differential equation

$-6\lambda\sin 3x - \cancel{9\lambda x\cos 3x} + \cancel{9\lambda x\cos 3x} = -12\sin 3x$

Hence

$\lambda = 2$

b The auxiliary equation is

$m^2 + 9 = 0 \Rightarrow m^2 = -9$

$m = \pm 3i$

The complementary function is given by

$y = A\cos 3x + B\sin 3x$

The general solution is

$y = A\cos 3x + B\sin 3x + 2x\cos 3x$

Part **a** shows that $2x\cos 3x$ is a particular integral of the differential equation and general solution = complementary function + particular integral

16 c $y = 1,\ x = 0$

$1 = A$

$\frac{dy}{dx} = -3A\sin 3x + 3B\cos 3x + 2\cos 3x - 6x\sin 3x$

Differentiate the general solution in part **b** with respect to x.

$\frac{dy}{dx} = 2,\ x = 0$

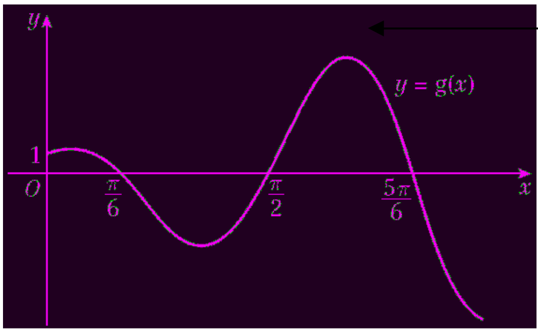
$2 = 3B + 2 \Rightarrow B = 0$

The particular solution is

$y = \cos 3x + 2x\cos 3x = (1 + 2x)\cos 3x$

d For $x > 0$, the curve crosses the x-axis at $\cos 3x = 0$

$3x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \Rightarrow x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$



The boundary conditions give you that at $x = 0,\ y = 1$ and the curve has a positive gradient. The curve must then turn down and cross the axis at the three points where $x = \frac{\pi}{6}, \frac{\pi}{2}$ and $\frac{5\pi}{6}$.
The $(1 + 2x)$ factor in the general solution means that the size of the oscillations increases as x increases.

17 a If $y = Kt^2 e^{3t}$

$\frac{dy}{dt} = 2Kt\,e^{3t} + 3Kt^2\,e^{3t}$

$\frac{d^2y}{dt^2} = 2K\,e^{3t} + 3Kt^2\,e^{3t} + 6Kt\,e^{3t} + 9Kt^2\,e^{3t}$

$= 2K\,e^{3t} + 12Kt\,e^{3t} + 9Kt^2\,e^{3t}$

Substituting into the differential equation

e^{3t} cannot be zero, so you can divide throughout by e^{3t}.

$2K\,e^{3t} + \cancel{12Kt\,e^{3t}} + \cancel{9Kt^2\,e^{3t}} - \cancel{12Kt\,e^{3t}} - \cancel{18Kt\,e^{3t}} + \cancel{9Kt\,e^{3t}} = 4\,e^{3t}$

Hence

$2K = 4 \Rightarrow K = 2$

$2t^2\,e^{3t}$ is a particular integral of the differential equation.

17 b The auxiliary equation is

$$m^2 - 6m + 9 = 0$$
$$(m-3)^2 = 0$$
$$m = 3, \text{ repeated}$$

The complementary function is given by

$$y = e^{3t}(A + Bt)$$

If the auxiliary equation has a repeated root α, then the complementary function is $e^{\alpha t}(A + Bt)$. You can quote this result.

The general solution is

$$y = e^{3t}(A + Bt) + 2t^2 e^{3t} = (A + Bt + 2t^2)e^{3t}$$

c $y = 3,\ t = 0$

$$3 = A$$

$$\frac{dy}{dt} = (B + 4t)e^{3t} + 3(A + Bt + 2t^2)e^{3t}$$

$$\frac{dy}{dt} = 1,\ t = 0$$

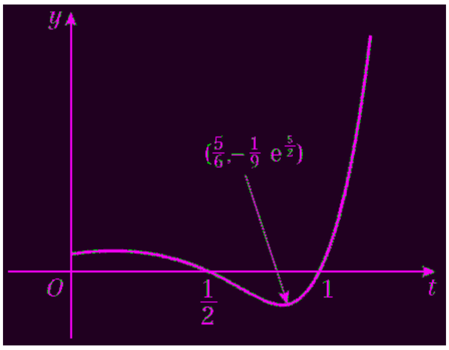
$$1 = B + 3A \Rightarrow B = 1 - 3A \Rightarrow B = -8$$

As $A = 3$.

The particular solution is $y = (3 - 8t + 2t^2)e^{3t}$

d This particular solution crosses the t-axis where

$$1 - 3t + 2t^2 = (1-2t)(1-t) = 0$$
$$t = \tfrac{1}{2}, 1$$



For a minimum $\frac{dy}{dt} = 0$

$$(-3 + 4t)e^{3t} + (1 - 3t + 2t^2)3e^{3t} = 0$$

e^{3t} cannot be zero, so you can divide throughout by e^{3t}.

$$-3 + 4t + 3 - 9t + 6t^2 = 0$$
$$6t^2 - 5t = t(6t - 5) = 0 \Rightarrow t = 0,\ \tfrac{5}{6}$$

From the diagram $t = \frac{5}{6}$ gives the minimum.

It is clear from the diagram that there is a minimum point between $t = \frac{1}{2}$ and $t = 1$.
You do not have to consider the second derivative to show that it is a minimum.

At $t = \frac{5}{6}$

$$y = \left(1 - 3 \times \tfrac{5}{6} + 2 \times \left(\tfrac{5}{6}\right)^2\right)e^{3 \times \frac{5}{6}} = -\tfrac{1}{9}e^{\frac{5}{2}}$$

The coordinates of the minimum point are

$\left(\frac{5}{6}, -\frac{1}{9}e^{\frac{5}{2}}\right)$.

18 a The auxiliary equation is

$$2m^2 + 5m + 2 = 0$$
$$(2m+1)(m+2) = 0$$
$$m = -\tfrac{1}{2}, -2$$

The complementary function is given by

$$x = Ae^{-\frac{1}{2}t} + Be^{-2t}$$

If the auxiliary equation has two real solutions α and β, the complementary function is $x = Ae^{\alpha t} + Be^{\beta t}$. You can quote this result.

For a particular integration, try $x = pt + q$

If the right hand side of the differential equation is a polynomial of degree n, then you can try a particular integral of the same degree. Here the right-hand side is linear, so you try the general linear function $pt + q$.

$$\frac{dx}{dt} = p, \quad \frac{d^2x}{dt^2} = 0$$

Substituting into the differential equation

$$0 + 5p + 2pt + 2q = 2t + 9$$

Equating the coefficients of t

$$2p = 2 \Rightarrow p = 1$$

Equating the constant coefficients

$$5p + 2q = 9 \Rightarrow q = \frac{9-5p}{2} \Rightarrow q = 2$$

A particular integral is $t + 2$

The general solution is

$$x = Ae^{-\frac{1}{2}t} + Be^{-2t} + t + 2$$

b $x = 3,\ t = 0$

$$3 = A + B + 2 \Rightarrow A + B = 1 \qquad \textbf{(1)}$$

$$\frac{dx}{dt} = -\tfrac{1}{2}Ae^{-\frac{1}{2}t} - 2Be^{-2t} + 1$$

Differentiating the general solution in part **a**.

$$\frac{dx}{dt} = -1,\ t = 0$$

$$-1 = -\tfrac{1}{2}A - 2B + 1 \Rightarrow \tfrac{1}{2}A + 2B = 2 \qquad \textbf{(2)}$$
$$A + 4B = 4 \qquad \textbf{(3)}$$
$$3B = 3 \Rightarrow B = 1$$

Multiplying (**2**) by 2 and subtracting (**1**) from (**3**).

Substituting $B = 1$ into (**1**)

$$A + 1 = 1 \Rightarrow A = 0$$

The particular solution is

$$x = e^{-2t} + t + 2$$

19 a If $x = At^2 e^{-t}$

$$\frac{dx}{dt} = 2At e^{-t} - At^2 e^{-t}$$

$$\frac{d^2x}{dt^2} = 2A e^{-t} - 2At e^{-t} - 2At e^{-t} + At^2 e^{-t}$$

$$= 2A e^{-t} - 4At e^{-t} + At^2 e^{-t}$$

Substituting into the differential equation

$$2A e^{-t} - \cancel{4At e^{-t}} + \cancel{At^2 e^{-t}} + \cancel{4At^2 e^{-t}} - \cancel{2At^2 e^{-t}} + \cancel{At^2 e^{-t}} = e^{-t}$$

e^{-t} cannot be zero, so you can divide throughout by e^{-t}.

Hence

$$2A = 1 \Rightarrow A = \tfrac{1}{2}$$

b The auxiliary equation is

$$m^2 + 2m + 1 = (m+1)^2 = 0$$

$$m = -1, \text{ repeated}$$

The complementary function is given by

$$x = e^{-t}(A + Bt)$$

If the auxiliary equation has a repeated root α, then the complementary function is $e^{\alpha t}(A + Bt)$. You can quote this result.

The general solution is

$$x = e^{-t}(A + Bt) + \tfrac{1}{2}t^2 e^{-t} = \left(A + Bt + \tfrac{1}{2}t^2\right)e^{-t}$$

$$x = 1,\ t = 0$$

From part **a**, $\frac{1}{2}t^2 e^{-t}$ is a particular integral of the differential equation.

$$1 = A$$

$$\frac{dx}{dt} = (B + t)e^{-t} - \left(A + Bt + \tfrac{1}{2}t^2\right)e^{-t}$$

$$\frac{dx}{dt} = 0,\ t = 0$$

$$0 = B - A \Rightarrow B = A = 1$$

The particular solution is

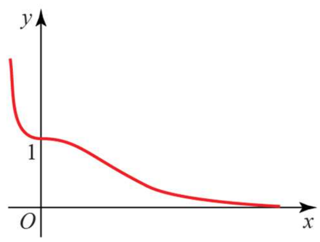
$$x = \left(1 + t + \tfrac{1}{2}t^2\right)e^{-t}$$

c

$$\frac{dx}{dt} = (1 + t)e^{-t} - \left(1 + t + \tfrac{1}{2}t^2\right)e^{-t}$$

$$= -\tfrac{1}{2}t^2 e^{-t} \leqslant 0, \text{for all real } t.$$

When $t = 0$, $x = 1$ and x has a negative gradient for all positive t, x is a decreasing function of t. Hence, for $t \geqslant 0$, $x \leqslant 1$, as required.



The graph of x against t, shows the curve crossing the x-axis at $x = 1$ and then decreasing. For all positive t, x is less than 1.

20 a $y = kx \Rightarrow \frac{dy}{dx} = k \Rightarrow \frac{d^2y}{dx^2} = 0$

Substituting into $\frac{d^2y}{dx^2} + y = 3x$

$$0 + kx = 3x$$
$$k = 3$$

b The auxiliary equation is

$m^2 + 1 = 0 \Rightarrow m = \pm i$

The complementary function is given by

$y = A\sin x + B\cos x$

and the general solution is

$y = A\sin x + B\cos x + 3x$

$y = 0,\ x = 0$

$0 = B + 0 \Rightarrow B = 0$

The most general solution is

$y = A\sin x + 3x$

In part **b**, only one condition is given, so only one of the arbitrary constants can be found. The solution is a family of functions, some of which are illustrated in the diagram below.

c At $x = \pi$

$y = A\sin\pi + 3\pi = 3\pi$

This is independent of the value of A.

Hence, all curves given by the solution in part **a** pass through $(\pi, 3\pi)$.

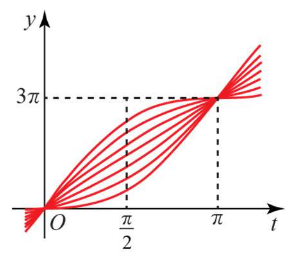
$\frac{dy}{dx} = A\cos x + 3$

At $x = \frac{\pi}{2}$

$\frac{dy}{dx} = A\cos\frac{\pi}{2} + 3 = 3$

This is independent of the value of A.

Hence, all curves given by the solution in part **a** have an equal gradient of 3 at $x = \frac{\pi}{2}$.



As is illustrated by this diagram, the family of curves $y = A\sin x + 3x$ all go through (0, 0) and $(\pi, 3\pi)$. The tangent to the curves at $x = \frac{\pi}{2}$ are all parallel to each other.

d $y = 0,\ x = \frac{\pi}{2}$

Substituting into $y = A\sin x + 3x$

$0 = A\sin\frac{\pi}{2} + \frac{3\pi}{2} = A + \frac{3\pi}{2} \Rightarrow A = -\frac{3\pi}{2}$

The particular solution is

$y = 3x - \frac{3\pi}{2}\sin x$

20 e For a minimum

$$\frac{dy}{dx} = 3 - \frac{3\pi}{2}\cos x = 0$$

$$\cos x = \frac{2}{\pi} \Rightarrow x = \arccos\left(\frac{2}{\pi}\right)$$

$$\frac{d^2y}{dx^2} = \frac{3\pi}{2}\sin x$$

In the interval $0 \leqslant x \leqslant \frac{\pi}{2}$,

$$\frac{d^2y}{dx^2} > 0 \Rightarrow \text{minimum}$$

$$\sin^2 x = 1 - \cos^2 x = 1 - \frac{4}{\pi^2} = \frac{\pi^2 - 4}{\pi^2}$$

In the interval $0 \leqslant x \leqslant \frac{\pi}{2}$

$$\sin x = +\left(\frac{\pi^2 - 4}{\pi^2}\right)^{\frac{1}{2}} = \frac{\sqrt{\pi^2 - 4}}{\pi}$$

$$y = 3\arccos\left(\frac{2}{\pi}\right) - \frac{3\pi}{2} \times \frac{\sqrt{\pi^2 - 4}}{\pi}$$

$$= 3\arccos\left(\frac{2}{\pi}\right) - \frac{3}{2}\sqrt{\pi^2 - 4}, \text{ as required.}$$

$\cos x = \frac{2}{\pi}$ has an infinite number of solutions. This shows that the solution in the first quadrant gives a minimum as $\sin x$ is positive in that quadrant.

21 a $y = \frac{1}{2}u - \frac{1}{2}x$

Differentiate throughout with respect to x

$$\frac{dy}{dx} = \frac{1}{2}\frac{du}{dx} - \frac{1}{2}$$

$$\frac{dy}{dx} = x + 2y$$

$y = \frac{1}{2}(u - x) \Rightarrow 2y = u - x$

transforms to

$$\frac{1}{2}\frac{du}{dx} - \frac{1}{2} = x + u - x = u$$

$$\frac{du}{dx} - 1 = 2u$$

$$\frac{du}{dx} = 2u + 1$$

This is a separable equation. You learnt how to solve separable equations in C4

$$\int \frac{1}{2u+1}\,du = \int 1\,dx$$

Separating the variables.

$$\frac{1}{2}\ln(2u+1) = x + A$$

$$\ln(2u+1) = 2x + B$$

Twice one arbitrary constant A is another arbitrary constant, $B = 2A$

$$e^{\ln(2u+1)} = e^{2x+B} = e^B e^{2x} = C\,e^{2x}$$

$$2u + 1 = 4y + 2x + 1 = C\,e^{2x}$$

e to an arbitrary constant is another arbitrary constant.

Here $C = e^B$

$$y = \frac{C\,e^{2x} - 2x - 1}{4}$$

This is the general solution of the original differential equation.

b $y = 2$ at $x = 0$

$$2 = \frac{C-1}{4} \Rightarrow 8 = C - 1 \Rightarrow C = 9$$

$$y = \frac{9e^{2x} - 2x - 1}{4}$$

This is the particular solution of the original differential equation for which $y = 2$ at $x = 0$

22 a $y = vx$

$$\frac{dy}{dx} = x\frac{dv}{dx} + v$$

Differentiating vx as a product, $\frac{d}{dx}(vx) = \frac{dv}{dx}x + v\frac{d}{dx}(x)$
$= x\frac{dv}{dx} + v$, as $\frac{d}{dx}(x) = 1$

Substituting $y = vx$ and $\frac{dy}{dx} = x\frac{dv}{dx} + v$ into

equation **(1)** in the question

$$x\frac{dv}{dx} + v = \frac{(4x+vx)(x+vx)}{x^2}$$
$$= \frac{x^2(4+v)(1+v)}{x^2} = (4+v)(1+v) = 4 + 5v + v^2$$

$$x\frac{dv}{dx} = 4 + 4v + v^2 = (2+v)^2, \text{ as required.}$$

This is a separable equation and the first step in its solution is to separate the variables, by collecting together the terms in v and dv on one side of the equation and the terms in x and dx on the other side of the equation.

b $$\int \frac{1}{(2+v)^2}\,dv = \int \frac{1}{x}\,dx$$

$$-\frac{1}{2+v} = \ln x + c$$

$$\int (2+v)^{-2}\,dv = \frac{(2+v)^{-1}}{-1} = -\frac{1}{2+v}$$

$$2 + v = -\frac{1}{\ln x + c}$$

$$v = -2 - \frac{1}{\ln x + c}$$

c $y = vx \Rightarrow v = \frac{y}{x}$

Substituting $v = \frac{y}{x}$ into the answer to part **b**

$$\frac{y}{x} = -2 - \frac{1}{\ln x + c}$$

$$y = -2x - \frac{x}{\ln x + c'} \text{ as required}$$

Multiply throughout by x to obtain the printed answer.

23 a $y = vx$

$$\frac{dy}{dx} = x\frac{dx}{dx} + v$$

Differentiating vx as a product,

$$\frac{d}{dx}(vx) = \frac{dv}{dx}x + v\frac{d}{dx}(x) = x\frac{dv}{dx} + v, \text{ as } \frac{d}{dx}(x) = 1$$

Substitute $y = vx$ and $\frac{dy}{dx} = x\frac{dv}{dx} + v$ into equation **(1)** in the question

$$x\frac{dv}{dx} + v = \frac{3x - 4vx}{4x + 3vx} = \frac{\cancel{x}(3 - 4v)}{\cancel{x}(4 + 3v)}$$

$$x\frac{dv}{dx} = \frac{3 - 4v}{4 + 3v} - v = \frac{3 - 4v - 4v - 3v^2}{4 + 3v} = \frac{3 - 8v - 3v^2}{4 + 3v}$$

$$x\frac{dv}{dx} = -\frac{3v^2 + 8v - 3}{3v + 4}, \text{ as required}$$

This is a separable equation and in part **b** you solve it by collecting together the terms in v and dv on one side of the equation and the terms in x and dx on the other side.

b $$\int \frac{3v + 4}{3v^2 + 8v - 3}dv = \frac{1}{2}\int \frac{6v + 8}{3v^2 + 8v - 3}dv = -\int \frac{1}{x}dx$$

$\int \frac{f'(x)}{f(x)}dx = \ln f(x)$ is a standard formula you should know. As $6v + 8$ is the derivative of $3v^2 + 8v - 3$,

$$\int \frac{6v + 8}{3v^2 + 8v - 3}dv = \ln(3v^2 + 8v - 3)$$

$$\frac{1}{2}\ln(3v^2 + 8v - 3) = -\ln x + A$$

$$\ln(3v^2 + 8v - 3) = -2\ln x + B$$

$$= \ln\frac{1}{x^2} + \ln C = \ln\frac{C}{x^2}$$

An arbitrary constant B can be written as the logarithm of another arbitrary constant $\ln C$.

Hence

$$3v^2 + 8v - 3 = \frac{C}{x^2}$$

c $y = xv \Rightarrow v = \frac{y}{x}$

Substituting into the answer to part **b**

$$\frac{3y^2}{x^2} + \frac{8y}{x} - 3 = \frac{C}{x^2}$$

$$3y^2 + 8yx - 3x^2 = C$$

Multiply each term in the equation by x^2

$y = 7$ at $x = 1$

$3 \times 49 + 56 - 3 = C \Rightarrow C = 200$

Factorising the left hand side of the equation

$(3y - x)(y + 3x) = 200$, as required.

24 a $\mu = y^{-2}$

$$\frac{d\mu}{dx} = -2 \times y^{-3} \times \frac{dy}{dx}$$

Differentiate both sides implicitly with respect to x

Hence

$$\frac{dy}{dx} = -\frac{y^3}{2}\frac{d\mu}{dx}$$

You transform this equation, making $\frac{dy}{dx}$ the subject of the formula as you need to substitute for $\frac{dy}{dx}$ in **(1)**

Substituting in equation **(1)** in the question

$$-\frac{y^3}{2}\frac{d\mu}{dx} - 2xy = xe^{-x^2}y^3$$

Divide by y^3

$$-\frac{1}{2}\frac{d\mu}{dx} + \frac{2x}{y^2} = xe^{-x^2}$$

As $\mu = \frac{1}{y^2}$

$$-\frac{1}{2}\frac{d\mu}{dx} + 2x\mu = xe^{-x^2}$$

Multiply by (–2)

$$\frac{d\mu}{dx} - 4x\mu = -2xe^{-x^2}$$, as required

24 b The integrating factor of **(2)** is

$$e^{\int -4x\,dx} = e^{-2x^2}$$

Multiplying **(2)** throughout by e^{-2x^2}

$$e^{-2x^2}\frac{d\mu}{dx} - 4x\mu e^{-2x^2} = -2x e^{-x^2} \times e^{-2x^2} = -2x e^{-3x^2}$$

$$\frac{d}{dx}(\mu e^{-2x^2}) = -2x e^{-3x^2}$$

$$\mu e^{-2x^2} = -2\int x e^{-3x^2}\,dx = \frac{1}{3}e^{-3x^2} + C$$

This integration can be carried out by inspection. As $\frac{d}{dx}(e^{-3x^2}) = -6x e^{-3x^2}$, then $\int x e^{-3x^2}dx = -\frac{1}{6}e^{-3x^2}$

Multiplying throughout by e^{2x^2}

$$\mu = \frac{1}{3}e^{-x^2} + C e^{2x^2}$$

c As $\mu = \frac{1}{y^2}$

$$\frac{1}{y^2} = \frac{1}{3}e^{-x^2} + C e^{2x^2}$$

$y = 1$ at $x = 0$

$$1 = \frac{1}{3} + C \Rightarrow C = \frac{2}{3}$$

$$\frac{1}{y^2} = \frac{1}{3}e^{-x^2} + \frac{2}{3}e^{2x^2}$$

As no form of the answer has been specified in the question, this is an acceptable answer for the particular solution of **(1)**

25 a $y = xv$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$$

Use the product rule for differentiation
$\frac{\mathrm{d}}{\mathrm{d}x}(xv) = \frac{\mathrm{d}}{\mathrm{d}x}(x)\times v + x\frac{\mathrm{d}v}{\mathrm{d}x} = 1\times v + x\frac{\mathrm{d}v}{\mathrm{d}x}$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}v}{\mathrm{d}x} + \frac{\mathrm{d}v}{\mathrm{d}x} + x\frac{\mathrm{d}^2v}{\mathrm{d}x^2} = 2\frac{\mathrm{d}v}{\mathrm{d}x} + x\frac{\mathrm{d}^2v}{\mathrm{d}x^2}$$

Substituting for $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ into **(1)**

$$x^2\left(x\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + 2\frac{\mathrm{d}v}{\mathrm{d}x}\right) - 2x\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) + (2+9x^2)vx = x^5$$

$$x^3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \cancel{2x^2\frac{\mathrm{d}v}{\mathrm{d}x}} - \cancel{2xv} - \cancel{2x^2\frac{\mathrm{d}v}{\mathrm{d}x}} + \cancel{2xv} + 9x^3v = x^5$$

$$x^3\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + 9x^3v = x^5$$

Divide by x^3

$$\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + 9v = x^2, \text{ as required}$$

25 b The auxiliary equation is

$$m^2+9=0 \Rightarrow m^2=-9$$
$$m=\pm\ 3\mathrm{i}$$

The complementary function is given by

$$v=A\cos 3x+B\sin 3x$$

For a particular integral, try $v=px^2+qx+r$

If the right hand side of the differential equation is a polynomial of degree n, then you can try a particular integral of the same degree. Here the right hand side is a quadratic x^2, so you try a general quadratic px^2+qx+r

$$\frac{\mathrm{d}v}{\mathrm{d}x}=2px+q,\ \frac{\mathrm{d}^2v}{\mathrm{d}x^2}=2p$$

Substituting into **(2)**

$$2p+9qx^2+9qx+9r=x^2$$

Equating coefficients of x^2

$$9p=1\Rightarrow q=\frac{1}{9}$$

Equating coefficient of x

$$9q=0\Rightarrow q=0$$

Equating constant coefficients

As $p=\frac{1}{9}$

$$2p+9r=0\Rightarrow 9r=-2p=-\frac{2}{9}\Rightarrow r=-\frac{2}{81}$$

A particular integral is $\frac{1}{9}x^2-\frac{2}{8!}$

A general solution of **(2)** is

$$v=A\cos 3x+B\sin 3x+\frac{1}{9}x^2-\frac{2}{81}$$

$y=vx\Rightarrow v=\frac{y}{x}$

c $\frac{y}{x}=A\cos 3x+B\sin 3x+\frac{1}{9}x^2-\frac{2}{81}$

The question does not ask for a particular form of the answer in part **c**, so this would be an acceleration answer.

$$y=Ax\cos 3x+Bx\sin 3x+\frac{1}{9}x^3-\frac{2}{81}x$$

26 a $x = t^{\frac{1}{2}} \Rightarrow \frac{dx}{dt} = \frac{1}{2}t^{-\frac{1}{2}} = \frac{1}{2t^{\frac{1}{2}}}$

$\frac{dt}{dx} = \frac{1}{\frac{1}{2t^{\frac{1}{2}}}} = 2t^{\frac{1}{2}}$

Use $\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dt} \times 2t^{\frac{1}{2}} = 2t^{\frac{1}{2}}\frac{dy}{dt}$

You obtain an expression for $\frac{dy}{dx}$ using the chain rule.

b Substituting $x = t^{\frac{1}{2}}$, the result of part **a** and the given $\frac{d^2y}{dx^2} = 4t\frac{d^2y}{dt^2} + 2\frac{dy}{dt}$ into **(1)**

$\left(6t^{\frac{1}{2}} - \frac{1}{t^{\frac{1}{2}}}\right) \times 2t^{\frac{1}{2}} = 6t^{\frac{1}{2}} \times 2t^{\frac{1}{2}} - \frac{2t^{\frac{1}{2}}}{t^{\frac{1}{2}}} = 12t - 2$

$4t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + \left(6t^{\frac{1}{2}} - \frac{1}{t^{\frac{1}{2}}}\right)2t^{\frac{1}{2}}\frac{dy}{dt} - 16ty = 4t\,e^{2t}$

$4t\frac{d^2y}{dt^2} + \cancel{2\frac{dy}{dt}} + 12t\frac{dy}{dt} - \cancel{2\frac{dy}{dt}} - 16ty = 4t\,e^{2t}$

$4t\frac{d^2y}{dt^2} + 12t\frac{dy}{dt} - 16ty = 4t\,e^{2t}$

Divide throughout by $4t$

$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 4y = e^{2t}$, as required

c The auxiliary equation is

$m^2 + 3m - 4 = (m-1)(m+4) = 0$

$m = 1, -4$

The complementary function is

$y = Ae^t + Be^{-4t}$

For a particular integral try, $y = k\,e^{2t}$

If the right hand side of the equation is $e^{\alpha t}$, you can try $k\,e^{\alpha t}$ as a particular integral. This will work unless α is a solution of the auxiliary equation.

$\frac{dy}{dt} = 2k\,e^{2t}$, $\frac{d^2y}{dt^2} = 4k\,e^{2t}$

Substituting into $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 4y = e^{2t}$

$\cancel{4k\,e^{2t}} + 6k\,e^{2t} \cancel{- 4k\,e^{2t}} = e^{2t}$

As e^{2t} cannot be zero, you can divide throughout by e^{2t}

$6k = 1 \Rightarrow k = \frac{1}{6}$

A particular integral is $\frac{1}{6}e^{2t}$

The general solution of the differential equation in y and t is

$y = Ae^t + Be^{-4t} + \frac{1}{6}e^{2t}$

$x = t^{\frac{1}{2}} \Rightarrow t = x^2$

The general solution of **(1)** is

$y = Ae^{x^2} + Be^{-4x^2} + \frac{1}{6}e^{2x^2}$

27 a $x = \ln t \Rightarrow \dfrac{dx}{dt} = \dfrac{1}{t} \Rightarrow \dfrac{dt}{dx} = t$

$\dfrac{dt}{dx} = \dfrac{1}{\frac{dx}{dt}}$

$\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx} = \dfrac{dy}{dt} \times t$

$\dfrac{dy}{dx} = t\dfrac{dy}{dt}$

b $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{dt}{dx} \times \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)$

$= t\dfrac{d}{dt}\left(t\dfrac{dy}{dt}\right)$

$= t\left(\dfrac{dy}{dt} + t\dfrac{d^2y}{dt^2}\right)$

$= t^2\dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt}$, as required

It is a common error to proceed from $\dfrac{dy}{dx} = t\dfrac{dy}{dt}$ to $\dfrac{d^2y}{dx^2} = \dfrac{dy}{dt} + t\dfrac{d^2y}{dt^2}$

This is incorrect because the left hand side has been differentiated with respect to x and the right hand side with respect to t. The version of the chain rule given here must be used.

c Substituting $x = \ln t, \dfrac{dy}{dx} = t\dfrac{dy}{dt}$ and

$\dfrac{d^2y}{dx^2} = t^2\dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt}$ into **(1)**

$e^{2\ln t} = e^{\ln t^2} = t^2$, using the log role $n \ln a = \ln a^n$ and $e^{\ln f(t)} = f(t)$

$t^2\dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt} - (1-6t)t\dfrac{dy}{dt} + 10yt^2 = 5t^2 \sin 2t$

$t^2\dfrac{d^2y}{dt^2} + \cancel{t\dfrac{dy}{dt}} - \cancel{t\dfrac{dy}{dt}} + 6t^2\dfrac{dy}{dt} + 10yt^2 = 5t^2 \sin 2t$

After cancelling, divide throughout by t^2

$\dfrac{d^2y}{dt^2} + 6\dfrac{dy}{dt} + 10y = 5\sin 2t$, as required

27 d The auxiliary equation of **(1)** is

$$m^2+6m+10=0$$
$$m^2+6m+9=-1$$
$$(m+3)^2=-1$$
$$m+3=\pm\text{i}$$
$$m=-3\pm\text{i}$$

The complementary function is given by
$y=\text{e}^{-3t}(A\cos t+B\sin t)$

For a particular integral try $y=p\sin 2t+q\cos 2t$

$$\frac{\text{d}y}{\text{d}x}=2p\cos 2t-2p\sin 2t$$
$$\frac{\text{d}^2y}{\text{d}x^2}=-4p\sin 2t-4q\cos 2t$$

If the right hand side of the second order differential equation is a k sin nt or k cos nt function, then you should try a particular integral of the form $p\cos nt+q\sin nt$

Substituting into **(2)**

$$-4p\sin 2t-4q\cos 2t+12p\cos 2t-12q\sin 2t+10p\sin 2t+10q\cos 2t=5\sin 2t$$
$$(-4q-12q+10p)\sin 2t+(-4q+12q+10q)\cos 2t=5\sin 2t$$
$$(6p-12q)\sin 2t+(12p+6q)\cos 2t=5\sin 2t$$

Equating the coefficients of sin $2t$

$6p-12q=5$ **(3)**
$12p+6q=0$ **(4)**

You can solve the simultaneous equations by any appropriate method.

From **(4)** $p=-\frac{6}{12}q=-\frac{1}{2}q$

Substitute into **(3)**

$$-3q-12q=-15q=5\Rightarrow q=-\frac{1}{3}$$

Hence $p=-\frac{1}{2}q=-\frac{1}{2}\times-\frac{1}{3}=\frac{1}{6}$

The general solution of **(2)** is

$$y=\text{e}^{-3t}(A\cos t+B\sin t)+\frac{1}{6}\sin 2t-\frac{1}{3}\cos 2t$$

$x=\ln t\Rightarrow t=\text{e}^x$

The general solution of **(1)** is

$$y=\text{e}^{-3\text{e}^x}(A\cos(\text{e}^x)+B\sin(\text{e}^x))+\frac{1}{6}\sin(2\text{e}^x)-\frac{1}{3}\cos(2\text{e}^x)$$

28 $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$

$= 1 - \frac{x^2}{2}$, neglecting terms in x^3 and higher powers

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$

$= x$, neglecting terms in x^3 and higher powers

The series of cos x and sin x are both given in the formulae book and may be quoted without proof, unless the question specifically asks for a proof.

$$11\sin x - 6\cos x + 5 = 11x - 6\left(1 - \frac{x^2}{2}\right) + 5$$
$$= 11x - 6 + 3x^2 + 5$$
$$= -1 + 11x + 3x^2$$

You substitute the abbreviated series into the expression and collect together terms.

$A = -1,\ B = 11,\ C = 3$

29 LHS $= \ln(x^2 - x + 1) + \ln(x + 1) - 3\ln x$

$= \ln[(x^2 - x + 1)(x + 1)] - \ln x^3$

$= \ln\left(\frac{x^3 + 1}{x^3}\right) = \ln\left(1 + \frac{1}{x^3}\right)$

You collect together the three terms of the left hand side (LHS) of the expression into a single logarithm using all three log rules; $\log x + \log y = \log xy$

$\log x - \log y = \log\left(\frac{x}{y}\right)$,

and $n\log x = \log x^n$.

$(x^2 - x + 1)(x + 1) = x^3 + x^2 - x^2 - x + x + 1$

$= x^3 + 1$

Substituting $\frac{1}{x^3}$ for x and n for r in the series

This series is given in the formulae booklet. It is valid for $-1 < x \leqslant 1$ and, if $x > 1$, then $0 < \frac{1}{x^3} < 1$ so the series is valid for this question.

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots + \frac{(-1)^{r+1}x^r}{r} + \ldots$$

LHS $= \frac{1}{x^3} - \frac{1}{2x^6} + \ldots + \frac{(-1)^{n-1}}{nx^{3n}} + \ldots$, as required

$(-1)^{n+1} = (-1)^{n-1}$. If n is odd, both sides are 1. If n is even, both sides are –1.

30 $e^{-2x} = 1 + (-2x) + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \dots$

$= 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \dots$

Substituting $-2x$ for x in the formula $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ and ignoring terms in x^4 and higher powers.

$\cos 5x = 1 - \frac{(5x)^2}{2!} + \dots$

$= 1 - \frac{25}{2}x^2 + \dots$

Substituting $5x$ for x in the formula $\cos x = 1 - \frac{x^2}{2!} + \dots$ and ignoring terms in x^4 and higher powers.

$e^{-2x}\cos 5x = \left(1 - 2x + 2x^2 - \frac{4}{3}x^3 + \dots\right)\left(1 - \frac{25}{2}x^2 + \dots\right)$

$= 1 - \frac{25}{2}x^2 - 2x + 25x^3 + 2x^2 - \frac{4}{3}x^3 + \dots$

$= 1 - 2x + \left(-\frac{25}{2} + 2\right)x^2 + \left(25 - \frac{4}{3}\right)x^3 + \dots$

$= 1 - 2x - \frac{21}{2}x^2 + \frac{71}{3}x^3 + \dots$

When multiplying out the brackets, you discard terms in x^4 and higher powers. For example, multiplying $2x^2$ by $-\frac{25}{2}x^2$ gives $-25x^4$ and you just ignore his term.

$A = 1,\ B = -2,\ C = -\frac{21}{2}, D = \frac{71}{3}$

31 a $(2x+3)^{-1} = 3^{-1}\left(1 + \frac{2x}{3}\right)^{-1}$

Part **a** is a binomial series with a rational index.

$= \frac{1}{3}\left(1 - \frac{2x}{3} + \frac{(-1)(-2)}{2.1}\left(\frac{2x}{3}\right)^2 + \frac{(-1)(-2)(-3)}{3.2.1}\left(\frac{2x}{3}\right)^3 + \dots\right)$

$= \frac{1}{3}\left(1 - \frac{2}{3}x + \frac{4}{9}x^2 - \frac{8}{27}x^3 + \dots\right)$

$= \frac{1}{3} - \frac{2}{9}x + \frac{4}{27}x^2 - \frac{8}{81}x^3 + \dots$

b $\frac{\sin 2x}{3+2x} = \sin 2x(3+2x)^{-1}$

$= \left(2x - \frac{(2x)}{3!} + \dots\right)\left(\frac{1}{3} - \frac{2}{9}x + \frac{4}{27}x^2 - \frac{8}{81}x^3 + \dots\right)$

$= \left(2x - \frac{4}{3}x^3 + \dots\right)\left(\frac{1}{3} - \frac{2}{9}x + \frac{4}{27}x^2 - \frac{8}{81}x^3 + \dots\right)$

$= \frac{2}{3}x - \frac{4}{9}x^2 + \frac{8}{27}x^3 - \frac{16}{81}x^4 - \frac{4}{9}x^3 + \frac{8}{27}x^4 + \dots$

$= \frac{2}{3}x - \frac{4}{9}x^2 + \left(\frac{8}{27} - \frac{4}{9}\right)x^3 + \left(\frac{8}{27} - \frac{16}{81}\right)x^4 + \dots$

$= \frac{2}{3}x - \frac{4}{9}x^2 - \frac{4}{27}x^3 + \frac{8}{81}x^4 + \dots$

When multiplying out the brackets, you discard terms in x^4 and higher powers. For example, multiplying $-\frac{4}{3}x^3$ by $\frac{4}{27}x^2$ gives $-\frac{16}{81}x^5$ and you ignore this term.

32 a $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$

$= 1 + \left(-\frac{x^2}{2} + \frac{x^4}{24}\right)$ **(1),**

neglecting terms above x^4

The expression $-\frac{x^2}{2!} + \frac{x^4}{4!}$ is used to replace the x in the standard series for $\ln(1+x)$.

$\ln(1+x) = x - \frac{x^2}{2} + \dots$

Using the expansion **(1)**

$$\ln(\cos x) = \ln\left(1 + \left(-\frac{x^2}{2} + \frac{x^4}{24}\right)\right)$$

$$= \left(-\frac{x^2}{2} + \frac{x^4}{24}\right) - \frac{1}{2}\left(-\frac{x^2}{2} + \frac{x^4}{24}\right)^2 + \dots$$

$$= -\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^4}{8} + \dots$$

$$= -\frac{x^2}{2} - \frac{x^4}{12} - \dots$$

$-\frac{1}{2}\left(-\frac{x^2}{2} + \frac{x^4}{24}\right)^2 = -\frac{x^4}{8} + \frac{x^6}{48} - \frac{x^8}{1152}$

but, as the expansion is only required up to the term in x^4, you only need the first of the three terms.

b $\ln(\sec x) = \ln\left(\frac{1}{\cos x}\right) = \ln 1 - \ln \cos x$

$= -\ln \cos x$

Using the log rule $\log\left(\frac{a}{b}\right) = \log a - \log b$ and the fact that $\ln 1 = 0$.

Using the result to part **a**

$$\ln(\sec x) = -\left(-\frac{x^2}{2} - \frac{x^4}{12} - \dots\right) = \frac{x^2}{2} + \frac{x^4}{12} + \dots$$

33 a Let $u = 1 + \cos 2x$, then $f(x) = \ln u$

$$\frac{du}{dx} = -2\sin 2x$$

$$f'(x) = f'(u)\frac{du}{dx} = \frac{1}{u}\frac{du}{dx} = \frac{1}{1+\cos 2x} \times -2\sin 2x$$

$$= \frac{-4\sin x \cos x}{2\cos^2 x}$$

$$= \frac{-2\sin x}{\cos x} = -2\tan x, \text{ as required}$$

Using the identities $\sin 2x = 2\sin x \cos x$ and $\cos 2x = 2\cos^2 x - 1$.

33 b $\mathrm{f}''(x) = -2\sec^2 x$

$\mathrm{f}'''(x) = -4\sec^2 x \tan x$

$\mathrm{f}''''(x) = -8\sec x \cdot \sec x \tan x \cdot \tan x - 4\sec^2 x \cdot \sec^2 x$

$= -8\sec^2 x \tan^2 x - 4\sec^4 x$

$= -\left(-4\sec^2 x \tan x \times -2\tan x + (-2\sec^2 x)^2\right)$

$= -\left(\mathrm{f}'''(x)\mathrm{f}'(x) + (\mathrm{f}''(x))^2\right)$, as required

$\mathrm{f}''''(x)$ is a symbol used for the fourth derivative of f(x) with respect to x. The symbol $\mathrm{f}^{(\mathrm{iv})}(x)$ is also used for the fourth derivative.

You use the product rule for differentiation $\frac{\mathrm{d}}{\mathrm{d}x}(uv) = v\frac{\mathrm{d}u}{\mathrm{d}x} + u\frac{\mathrm{d}v}{\mathrm{d}x}$ with $u = -4\sec^2 x$ and $v = \tan x$. You also use the chain rule

$\frac{\mathrm{d}}{\mathrm{d}x}(\sec^2 x) = 2\sec x \frac{\mathrm{d}}{\mathrm{d}x}(\sec x)$

$= 2\sec x \times \sec x \tan x.$

c $\mathrm{f}(0) = \ln(1 + \cos 0) = \ln 2$

$\mathrm{f}'(0) = -2\tan 0 = 0$

$\mathrm{f}''(0) = -2\sec^2 0 = -2$

$\mathrm{f}'''(0) = -4\sec^2 0 \tan 0 = 0$

$\mathrm{f}''''(0) = -\left(\mathrm{f}'''(0)\mathrm{f}'(0) + (\mathrm{f}''(0))^2\right)$

$= -\left(0 \times 0 + (-2)^2\right) = -4$

Using the result for part **b**.

$$\mathrm{f}(x) = \mathrm{f}(0) + x\mathrm{f}'(0) + \frac{x^2}{2!}\mathrm{f}''(0) + \frac{x^3}{3!}\mathrm{f}'''(0) + \frac{x^4}{4!}\mathrm{f}^{(\mathrm{iv})}(0) + \ldots$$

$$= \ln + 2 + x \times 0 + \frac{x^2}{2} \times -2 + \frac{x^3}{6} \times 0 + \frac{x^4}{24} \times -4 + \ldots$$

$$= \ln 2 - x^2 - \tfrac{1}{6}x^4 + \ldots$$

The formula for Maclaurin's series is given in the formulae booklet. For this question, you need the terms up to and including the term in x^4.

34 a Let $\mathrm{f}(x) = \cos 2x$ $\quad \mathrm{f}\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{2} = 0$

$\mathrm{f}'(x) = -2\sin 2x$ $\quad \mathrm{f}'\left(\frac{\pi}{4}\right) = -2\sin\frac{\pi}{2} = -2$

$\mathrm{f}''(x) = -4\cos 2x$ $\quad \mathrm{f}''\left(\frac{\pi}{4}\right) = -4\cos\frac{\pi}{2} = 0$

$\mathrm{f}'''(x) = 8\sin 2x$ $\quad \mathrm{f}'''\left(\frac{\pi}{4}\right) = 8\sin\frac{\pi}{2} = 8$

$\mathrm{f}^{(\mathrm{iv})}(x) = 16\cos 2x$ $\quad \mathrm{f}^{(\mathrm{iv})}\left(\frac{\pi}{4}\right) = 16\cos\frac{\pi}{2} = 0$

$\mathrm{f}^{(\mathrm{v})}(x) = -32\sin 2x$ $\quad \mathrm{f}^{(\mathrm{v})}\left(\frac{\pi}{4}\right) = -32\sin\frac{\pi}{2} = -32$

Taylor's and Maclaurin's series need repeated differentiation and substitution. You need to display these in systematic form, both to help you substitute correctly and to show your working clearly so that the examiner can award you marks.

$\mathrm{f}^{(\mathrm{iv})}(x)$ and $\mathrm{f}^{(\mathrm{v})}(x)$ are symbols which can be used for the fourth and fifth derivatives of f(x) respectively.

$$\mathrm{f}(x) = \mathrm{f}(a) + (x-a)\mathrm{f}'(a) + \frac{(x-a)^2}{2!}\mathrm{f}''(a) + \frac{(x-a)^3}{3!}\mathrm{f}'''(a) + \frac{(x-a)}{4!}\mathrm{f}^{(\mathrm{iv})}(a) + \frac{(x-a)}{5!}\mathrm{f}^{(\mathrm{v})}(a) + \ldots$$

Substituting $\mathrm{f}(x) = \cos 2x$ and $a = \frac{\pi}{4}$

This is the appropriate form of Taylor's series for this question. It is given in the formula booklet.

$$\cos 2x = \left(x - \frac{\pi}{4}\right) \times (-2) + \frac{\left(x - \frac{\pi}{4}\right)^3}{6} \times 8 + \frac{\left(x - \frac{\pi}{4}\right)^5}{120} \times (-32) + \ldots$$

$$= -2\left(x - \frac{\pi}{4}\right) + \frac{4}{3}\left(x - \frac{\pi}{4}\right)^3 - \frac{4}{15}\left(x - \frac{\pi}{4}\right)^5 + \ldots$$

All of the even derivatives are zero at $x = \frac{\pi}{4}$

34 b Let $x = 1$, then $x - \frac{\pi}{4} = 0.2146\ldots$

Work out $x - \frac{\pi}{4}$ on your calculator and then use the ANS button to complete the calculation.

Substituting into the result of part **a**

$$\cos 2 = -2(0.2146\ldots) + \frac{4}{3}(0.2146\ldots)^3 - \frac{4}{15}(0.2146\ldots)^5 + \ldots$$
$$\approx -0.416147\,(6\,\text{d.p.})$$

This is a very accurate estimate and is correct to 6 decimal places.

35 a Let $f(x) = \ln(\sin x)$ $\quad f\left(\frac{\pi}{6}\right) = \ln\frac{1}{2} = -\ln 2$

$f'(x) = \frac{\cos x}{\sin x} = \cot x$ $\quad f'\left(\frac{\pi}{6}\right) = \cot\frac{\pi}{6} = \sqrt{3}$

$f''(x) = -\operatorname{cosec}^2 x$ $\quad f''\left(\frac{\pi}{6}\right) = -4$

$$\operatorname{cosec}\frac{\pi}{6} = \frac{1}{\sin\frac{\pi}{6}} = \frac{1}{\frac{1}{2}} = 2$$

$f'''(x) = 2\operatorname{cosec}^2 x \cot x$ $\quad f'''\left(\frac{\pi}{6}\right) = 2 \times 2^2 \times \sqrt{3} = 8\sqrt{3}$

Using the chain rule,
$$\frac{d}{dx}(-\operatorname{cosec}^2 x) = -2\operatorname{cosec} x \frac{d}{dx}(\operatorname{cosec} x)$$
$$= -2\operatorname{cosec} x \times -\operatorname{cosec} x \cot x$$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots$$

This is the appropriate form of Taylor's series for this question. It is given in the formula booklet.

Substituting $f(x) = \ln(\sin x)$ and $a = \frac{\pi}{6}$

$$\ln(\sin x) = -\ln 2 + \left(x - \frac{\pi}{6}\right) \times \sqrt{3} + \frac{1}{2}\left(x - \frac{\pi}{6}\right)^2 \times (-4) + \frac{1}{6}\left(x - \frac{\pi}{6}\right)^3 \times 8\sqrt{3} + \ldots$$

$$= -\ln 2 + \sqrt{3}\left(x - \frac{\pi}{6}\right) - 2\left(x - \frac{\pi}{6}\right)^2 + \frac{4\sqrt{3}}{3}\left(x - \frac{\pi}{6}\right)^3 + \ldots$$

Work out $x - \frac{\pi}{4}$ on your calculator and then use the ANS button to complete the calculation.

b Let $x = 0.5$, then $x - \frac{\pi}{6} = -0.0235987\ldots$

Substituting into the result of part **a**

$$\ln(\sin 0.5) = -\ln 2 + \sqrt{3}(-0.023598\ldots) - 2(-0.023598\ldots)^2 + \frac{4\sqrt{3}}{3}(-0.023\,598\ldots)^3 + \ldots$$
$$\approx -0.735166\ (6\ \text{d.p.})$$

36 a $y = \tan x$

$$\frac{dy}{dx} = \sec^2 x$$

$$\frac{d^2y}{dx^2} = 2\sec x \frac{d}{dx}(\sec x) = 2\sec x \times \sec x \tan x$$

$$= 2\sec^2 x \tan x$$

Using the chain rule for differentiation.

$$\frac{d^3y}{dx^3} = \tan x \frac{d}{dx}(2\sec^2 x) + 2\sec^2 x \frac{d}{dx}(\tan x)$$

$$= 4\sec^2 x \tan^2 x + 2\sec^4 x$$

Using the product rule for differentiation $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$ with $u = 2\sec^2 x$ and $v = \tan x$

b Let $y = f(x) = \tan x$

$$f\left(\frac{\pi}{4}\right) = \tan\frac{\pi}{4} = 1$$

Using the results in part **a**

$$f'\left(\frac{\pi}{4}\right) = \sec^2\frac{\pi}{4} = (\sqrt{2})^2 = 2$$

$$f''(x) = 2\sec^2\frac{\pi}{4}\tan\frac{\pi}{4} = 2\times(\sqrt{2})^2\times 1 = 4$$

$$f'''\left(\frac{\pi}{4}\right) = 2\sec^2\frac{\pi}{4}\tan^2\frac{\pi}{4} + 2\sec^4\frac{\pi}{4}$$

$$= 4(\sqrt{2})^2\times 1^2 + 2(\sqrt{2})^4 = 8 + 8 = 16$$

$\sec\frac{\pi}{4} = \sqrt{2}$ and $\tan\frac{\pi}{4} = 1$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots$$

This is the first four terms of Taylor's series.

Substituting $f(x) = \tan x$ and $x = \frac{\pi}{4}$

You are expanding $\tan x$ about the point $x = \frac{\pi}{4}$, using Taylor's series.

$$\tan x = 1 + \left(x - \frac{\pi}{4}\right)\times 2 + \frac{1}{2}\left(x - \frac{\pi}{4}\right)^2 \times 4 + \frac{1}{6}\left(x - \frac{\pi}{4}\right)^3 \times 16 + \ldots$$

$$= 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3 + \ldots$$

c Let $x = \frac{3\pi}{10}$, then $x - \frac{\pi}{4} = \frac{3\pi}{10} - \frac{\pi}{4} = \frac{\pi}{20}$

Substituting into the result in part **b**

$$\tan\frac{3\pi}{10} = 1 + 2\left(\frac{\pi}{20}\right) + 2\left(\frac{\pi}{20}\right)^2 + \frac{8}{3}\left(\frac{\pi}{20}\right)^3 + \ldots$$

$$\approx 1 + \frac{\pi}{10} + \frac{\pi^2}{200} + \frac{\pi^3}{3000}, \text{ as required.}$$

37 $f(x)=\ln x$, $f'(x)=\frac{1}{x}$, $f''(x)=-\frac{1}{x^2}$

$f'''(x)=\frac{2}{x^3}$

$\ln x = f(1)+f'(1)(x-1)+(x-1)^2\frac{f''(1)}{2!}+$

$(x-1)^3\frac{f'''(1)}{3!}+\ldots(x-1)^3\frac{f'''(1)}{3!}+\ldots$

$=0+(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3+\ldots$

38 a $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+2y=0$ **(1)**

Differentiate **(1)** throughout with respect to x

$-2x\frac{d^2y}{dx^2}+(1-x^2)\frac{d^3y}{dx^3}-\frac{dy}{dx}-x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$ **(2)**

Using the product rule for differentiation $\frac{d}{dx}(uv)=v\frac{du}{dx}+u\frac{dv}{dx}$ with $u=1-x^2$ and $v=\frac{d^2y}{dx^2}$, $\frac{d}{dx}\left((1-x^2)\frac{d^2y}{dx^2}\right)$

$=\frac{d^2y}{dx^2}\frac{d}{dx}(1-x^2)+(1-x^2)\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$

$=\frac{d^2y}{dx^2}\times -2x+(1-x^2)\frac{d^3y}{dx^3}$

Substituting $x=0$, $y=2$ and $\frac{dy}{dx}=-1$ into **(2)**

$0+\frac{d^3y}{dx^3}+1-0-2=0$

At $x=0$, $\frac{d^3y}{dx^3}=1$

38 b Let $y = \mathrm{f}(x)$

From the data in the question

$\mathrm{f}(0) = 2,\ \mathrm{f}'(0) = -1$

At $x = 0$, **(1)** above becomes

$\mathrm{f}''(0) + 2 \times 2 = 0 \Rightarrow \mathrm{f}''(0) = -4$

And the result to part **a** becomes

$\mathrm{f}'''(0) = 1$

$$\mathrm{f}(x) = \mathrm{f}(0) + x\mathrm{f}'(0) + \frac{x^2}{2!}\mathrm{f}''(0) + \frac{x^3}{3!}\mathrm{f}'''(0) + \ldots$$

The formula for Maclaurin's series is given in the formulae booklet. For this question, you need the terms up to and including the term in x^3

$$y = 2 + x \times (-1) + \frac{x^2}{2} \times (-4) + \frac{x^3}{6} \times 1 + \ldots$$
$$= 2 - x - 2x^2 + \frac{1}{6}x^3 + \ldots$$

39 a $$(1+2x)\frac{\mathrm{d}y}{\mathrm{d}x} = x + 4y^2 \quad *$$

You need to differentiate $4y^2$ implicitly with respect to x
$\frac{\mathrm{d}}{\mathrm{d}x}(4y^2) = \frac{\mathrm{d}y}{\mathrm{d}x} \times \frac{\mathrm{d}}{\mathrm{d}y}(4y^2) = 8y\frac{\mathrm{d}y}{\mathrm{d}x}$

Differentiate $*$ throughout with respect to x

$$2\frac{\mathrm{d}y}{\mathrm{d}x} + (1+2x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 1 + 8y\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$(1+2x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 1 + 8y\frac{\mathrm{d}y}{\mathrm{d}x} - 2\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$= 1 + 2(4y-1)\frac{\mathrm{d}y}{\mathrm{d}x} \quad \textbf{(1)} \text{ as required.}$$

b Differentiate **(1)** throughout with respect to x

$$2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (1+2x)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 8\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2(4y-1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\ldots \quad \textbf{(2)}$$

When using the product rule for differentiation $\frac{\mathrm{d}}{\mathrm{d}x}(uv) = v\frac{\mathrm{d}u}{\mathrm{d}x} + u\frac{\mathrm{d}v}{\mathrm{d}x}$ with $u = 2(4y-1)$ and $v = \frac{\mathrm{d}y}{\mathrm{d}x}$, $2(4y-1)$ must be differentiated implicitly with respect to x. So $\frac{\mathrm{d}}{\mathrm{d}x}\left(2(4y-1)\frac{\mathrm{d}y}{\mathrm{d}x}\right)$
$$= 8\frac{\mathrm{d}y}{\mathrm{d}x} \times \frac{\mathrm{d}y}{\mathrm{d}x} + 2(4y-1)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$$
$$= 8\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2(4y-1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$$

39 c Let $y = \mathrm{f}(x)$

From the data in the question

$$\mathrm{f}(0) = \frac{1}{2}$$

At $x = 0$, $y = \frac{1}{2}$, $*$ becomes

$$\mathrm{f}'(0) = 4 \times \left(\frac{1}{2}\right)^2 = 1$$

At $x = 0$, $y = \frac{1}{2}$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$, **(1)** becomes

$$\mathrm{f}''(0) = 1 + 2\left(4 \times \frac{1}{2} - 1\right) \times 1 = 3$$

At $x = 0$, $y = \frac{1}{2}$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$, $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 3$, **(2)** becomes

$$2 \times 3 + \mathrm{f}'''(0) = 8 \times 1^2 + 2\left(4 \times \frac{1}{2} - 1\right) \times 3$$

$$6 + \mathrm{f}'''(0) = 8 + 6 \Rightarrow \mathrm{f}'''(0) = 8$$

$$\mathrm{f}(x) = \mathrm{f}(0) + x\mathrm{f}'(0) + \frac{x^2}{2!}\mathrm{f}''(0) + \frac{x^3}{3!}\mathrm{f}'''(0) + \ldots$$

The formula for Maclaurin's series is given in the formulae booklet. For this question, you need the terms up to and including the term in x^3

$$y = \frac{1}{2} + x \times 1 + \frac{x^2}{2} \times 3 + \frac{x^3}{6} \times 8 + \ldots$$
$$= \frac{1}{2} + x + \frac{3}{2}x^2 + \frac{4}{3}x^3 + \ldots$$

40 a Let $y = \mathrm{f}(x)$

From the data in the question

$\mathrm{f}(0) = 1$

$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 + xy + x$ **(1)**

y^2 has to be differentiated implicitly by x. So $\frac{\mathrm{d}}{\mathrm{d}x}(y^2) = \frac{\mathrm{d}y}{\mathrm{d}x} \times \frac{\mathrm{d}}{\mathrm{d}y}(y^2) = \frac{\mathrm{d}y}{\mathrm{d}x} \times 2y$

At $x = 0$, $y = 1$, **(1)** becomes

$\mathrm{f}'(0) = 1^2 + 0 + 0 = 1$

Differentiating **(1)** throughout by x

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2y\frac{\mathrm{d}y}{\mathrm{d}x} + y + x\frac{\mathrm{d}y}{\mathrm{d}x} + 1$ **(2)**

Using the product rule for differentiation $\frac{\mathrm{d}}{\mathrm{d}x}(uv) = v\frac{\mathrm{d}u}{\mathrm{d}x} + u\frac{\mathrm{d}v}{\mathrm{d}x}$ with $u = x$ and $v = y$,

$\frac{\mathrm{d}}{\mathrm{d}x}(xy) = y\frac{\mathrm{d}x}{\mathrm{d}x} + x\frac{\mathrm{d}y}{\mathrm{d}x}$

$= y \times 1 + x\frac{\mathrm{d}y}{\mathrm{d}x}$

At $x = 0$, $y = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$, **(2)** becomes

$\mathrm{f}''(0) = 2 \times 1 \times 1 + 1 + 0 + 1 = 4$

Differentiate **(2)** throughout by x

$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}y}{\mathrm{d}x} + x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ **(3)**

Using the product rule for differentiation $\frac{\mathrm{d}}{\mathrm{d}x}(uv) = v\frac{\mathrm{d}u}{\mathrm{d}x} + u\frac{\mathrm{d}v}{\mathrm{d}x}$ with $u = 2y$ and $v = \frac{\mathrm{d}y}{\mathrm{d}x}$,

$\frac{\mathrm{d}}{\mathrm{d}x}\left(2y\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}(2y) + 2y\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$

$= \frac{\mathrm{d}y}{\mathrm{d}x} \times 2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$

At $x = 0$, $y = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4$, **(3)** becomes

$\mathrm{f}'''(0) = 2 \times 1^2 + 2 \times 1 \times 4 + 1 + 1 + 0 = 12$

$\mathrm{f}(x) = \mathrm{f}(0) + x\mathrm{f}'(0) + \frac{x^2}{2!}\mathrm{f}''(0) + \frac{x^3}{3!}\mathrm{f}'''(0) + \ldots$

$y = 1 + x \times 1 + \frac{x^2}{2} \times 4 + \frac{x^3}{6} \times 12 + \ldots$

$= 1 + x + 2x^2 + 2x^3 + \ldots$

b At 0.1

$y = 1 + 0.1 + 2(0.1)^2 + 2(0.1)^3 + \ldots$

$\approx 1 + 0.1 + 0.02 + 0.002 = 1.122$

$y \approx 1.12$ (2 d.p.)

41 a Rearranging the differential equation in the question

$$(y^2+y)\frac{dy}{dx}=x+3 \qquad \textbf{(1)}$$

The right hand side of the equation in the question would be hard to repeatedly differentiate as a quotient, so multiply both sides by $y+1$

Let $y=f(x)$

From the data in the question

$f(0)=1.5$

At $x=0,\ y=1.5$, **(1)** becomes

$$(1.5^2+1.5)f'(0)=0+3 \Rightarrow f'(0)=\frac{3}{3.75}=0.8$$

Differentiate **(1)** throughout by x

$$(2y+1)\left(\frac{dy}{dx}\right)^2+(y^2+y)\frac{d^2y}{dx^2}=1 \qquad \textbf{(2)}$$

Differentiating $\left(\frac{dy}{dx}\right)^2$ by x, using the chain rule

$$\frac{d}{dx}\left(\left(\frac{dy}{dx}\right)^2\right)=2\frac{dy}{dx}\times\frac{d}{dx}\left(\frac{dy}{dx}\right)$$
$$=2\frac{dy}{dx}\times\frac{d^2y}{dx^2}$$

At $x=0,\ y=1.5,\ \frac{dy}{dx}=0.8$, **(2)** becomes

$$4\times0.8^2+(1.5^2+1.5)f''(0)=1$$

$$f''(0)=\frac{1-4\times0.8^2}{3.75}=-0.416$$

Differentiate **(2)** throughout by x

$$2\left(\frac{dy}{dx}\right)^3+(2y+1)2\times\frac{dy}{dx}\times\frac{d^2y}{dx^2}+(2y+1)\frac{dy}{dx}\times\frac{d^2y}{dx^2}+(y^2+y)\frac{d^3y}{dx^3}=0$$

$$2\left(\frac{dy}{dx}\right)^3+3(2y+1)\frac{dy}{dx}\frac{d^2y}{dx^2}+(y^2+y)\frac{d^3y}{dx^3}=0 \qquad \textbf{(3)}$$

At $x=0,\ y=1.5,\ \frac{dy}{dx}=0.8,\ \frac{d^2y}{dx^2}=-0.416$, **(2)** becomes

$$2\times0.8^3+3\times4\times0.8\times-0.416+(1.5^2+1.5)f'''(0)=0$$

$$1.204-3.9936+3.75f'''(0)=0$$

$$f'''(0)=\frac{3.9936-1.204}{3.75}=0.79189\dot{3}$$

This is a recurring decimal. There is an exact fraction $\frac{7424}{9375}$

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots$$

$$y=1.5+x\times0.8+\frac{x^2}{2}\times-0.416+\frac{x^3}{6}\times0.79189\dot{3}+\ldots$$

$$=1.5+0.8x-0.208x^2+0.13198\dot{2}x^3+\ldots$$

b At $x=0.1$,

$$y=1.5+0.8(0.1)-0.208(0.1)^2+0.13198\dot{2}(0.1)^3+\ldots$$
$$=1.578\ldots$$

42 a $y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+y=0$ **(1)**

Differentiate **(1)** throughout with respect to x

$$\frac{dy}{dx}\times\frac{d^2y}{dx^2}+y\frac{d^3y}{dx^3}+2\frac{dy}{dx}\times\frac{d^2y}{dx^2}+\frac{dy}{dx}=0$$

$$y\frac{d^3y}{dx^3}=-3\frac{dy}{dx}\frac{d^2y}{dx^2}-\frac{dy}{dx}=-\frac{dy}{dx}\left(3\frac{d^2y}{dx^2}+1\right)$$

$$\frac{d^3y}{dx^3}=-\frac{1}{y}\frac{dy}{dx}\left(3\frac{d^2y}{dx^2}+1\right) \quad \textbf{(2)}$$

Using the product rule for differentiation $\frac{d}{dx}(uv)=v\frac{du}{dx}+u\frac{dv}{dx}$

with $u=y$ and $v=\frac{d^2y}{dx^2}$,

$$\frac{d}{dx}\left(y\frac{d^2y}{dx^2}\right)=\frac{d^2y}{dx^2}\times\frac{dy}{dx}+y\times\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$
$$=\frac{dy}{dx}\times\frac{d^2y}{dx^2}+y\frac{d^3y}{dx^3}$$

The wording of the question requires you to make $\frac{d^3y}{dx^3}$ the subject of the formula. There are many possible alternative forms for the answer.

b Let $y=f(x)$

From the data in the question

$f(0)=1,\ f'(0)=1$

At $x=0,\ y=1,\ \frac{dy}{dx}=1$, **(1)** becomes

$1\times f''(0)+1^2+1=0 \Rightarrow f''(0)=-2$

At $x=0,\ y=1,\ \frac{dy}{dx}=1,\ \frac{d^2y}{dx^2}=-2$, **(2)** becomes

$$f'''(0)=-\frac{1}{1}\times1(3\times-2+1)=-1(-6+1)=5$$

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots$$

$$y=1+x\times1+\frac{x^2}{2}\times-2+\frac{x^3}{6}\times5+\ldots$$

$$=1+x-x^2+\frac{5}{6}x^3+\ldots$$

The formula for Maclaurin's series is given in the formulae booklet. For this question, you need the terms up to and including the term in x^3

c The series expansion up to and including the term in x^3 can be used to estimate y if x is small. So it would be sensible to use it at $x=0.2$ but not at $x=50$

43 a $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 3y^2 = 6$ **(1)**

$3y^2$ has to be differentiated implicitly with respect to x

So $\dfrac{d}{dx}(3y^2) = \dfrac{dy}{dx} \times \dfrac{d}{dy}(3y^2) = \dfrac{dy}{dx} \times 6y$

Let $y = f(x)$

From the data in the question

$f(0) = 1,\ f'(0) = 0$

At $x = 0,\ y = 1,\ \dfrac{dy}{dx} = 0$, **(1)** becomes

$f''(0) - 4 \times 0 + 3 \times 1^2 = 6 \Rightarrow f''(0) = 3$

Differentiate **(1)** throughout with respect to x

$\dfrac{d^3y}{dx^3} - 4\dfrac{d^2y}{dx^2} + 6y\dfrac{dy}{dx} = 0$ **(2)**

At $x = 0,\ y = 1,\ \dfrac{dy}{dx} = 0,\ \dfrac{d^2y}{dx^2} = 3$, **(3)** becomes

$f'''(0) - 4 \times 3 + 6 \times 1 \times 0 = 0 \Rightarrow f'''(0) = 12$

Differentiate **(2)** throughout with respect to x

$\dfrac{d^4y}{dx^4} - 4\dfrac{d^3y}{dx^3} + 6\left(\dfrac{dy}{dx}\right)^2 + 6y\dfrac{d^2y}{dx^2} = 0$ **(3)**

Using the product rule for differentiation $\dfrac{d}{dx}(uv) = v\dfrac{du}{dx} + u\dfrac{dv}{dx}$

with $u = 6y$ and $v = \dfrac{dy}{dx}$,

$$\frac{d}{dx}\left(6y\frac{dy}{dx}\right) = \frac{dy}{dx}\frac{d}{dy}(6y) + 6y\frac{d}{dx}\left(\frac{dy}{dx}\right) = 6\left(\frac{dy}{dx}\right)^2 + 6y\frac{d^2y}{dx^2}$$

At $x = 0,\ y = 1,\ \dfrac{dy}{dx} = 0,\ \dfrac{d^2y}{dx^2} = 3,\ \dfrac{d^3y}{dx^3} = 12$,

(3) becomes

$f''''(0) - 4 \times 12 + 6 \times 0^2 + 6 \times 1 \times 3 = 0$

$f''''(0) = 48 - 18 = 30$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(iv)}(0) + \ldots$$

$$y = 1 + x \times 0 + \frac{x^2}{2} \times 3 + \frac{x^3}{6} \times 12 + \frac{x^4}{24} \times 30 + \ldots$$

$$= 1 + \frac{3}{2}x^2 + 2x^3 + \frac{5}{4}x^4 + \ldots$$

The formula for Maclaurin's series is given in the formulae booklet. For this question, you need the terms up to and including the term in x^4

b At $x = 0.2$

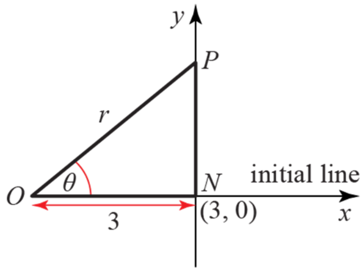
$y = 1 + 0.06 + 0.016 + 0.002 + \ldots$

$y \approx 1.08$ (2 d.p.)

44 a $r = 2$

You can just write the answer to part **a** down. The equation $r = k$ is the equation of a circle centre O and radius k, for any positive k.

b



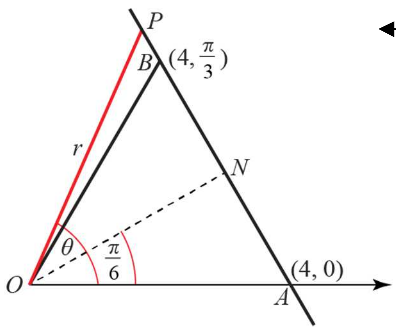
For any point P on the line

$$\frac{3}{r} = \cos\theta$$

$$r = \frac{3}{\cos\theta} = 3\sec\theta$$

If the point (3, 0) is labelled N, trigonometry on the right-angled triangle ONP gives the polar equation of the line.

c



In this diagram, the point (4, 0) is labelled A, the point $\left(4, \frac{\pi}{3}\right)$ is labelled B and the foot of the perpendicular from O to AB is labelled N. The triangle OAB is equilateral and $\angle AON = \frac{1}{2}60^\circ = 30^\circ = \frac{\pi}{6}$ radians.

In the triangle ONA

$$\frac{ON}{OA} = \frac{ON}{4} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$ON = 2\sqrt{3}$$

In the triangle ONP,

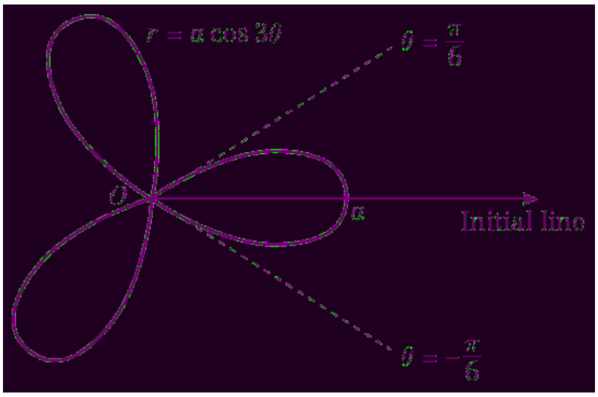
$$\frac{ON}{OP} = \cos\left(\theta - \frac{\pi}{6}\right)$$

$$\frac{2\sqrt{3}}{t} = \cos\left(\theta - \frac{\pi}{6}\right)$$

$$r = 2\sqrt{3}\sec\left(\theta - \frac{\pi}{6}\right)$$

This relation is true for any point P on the line and, as $OP = r$ this gives you the polar equation of the line.

45 a



At $\theta=-\frac{\pi}{6}$, $r=0$. As θ increases, r increases until $\theta=0$. For $\theta=0$, $a\cos 6\theta$ has its greatest value of a. Then, as θ increases, r decreases to 0 at $\theta=\frac{\pi}{6}$. Between $\theta=\frac{\pi}{6}$ and $\theta=\frac{\pi}{2}$, $\cos 6\theta$ is negative and, as $r\geqslant 0$, the curve does not exist. The pattern repeats itself in the other intervals where the curve exists.

b $A=\frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} r^2\,\mathrm{d}\theta$

Using $\cos 2A = 2\cos^2 A - 1$ with $A = 3\theta$.

$$\frac{1}{2}\int a^2\cos^2 3\theta\,\mathrm{d}\theta = \frac{a^2}{2}\int\left(\frac{1}{2}\cos 6\theta+\frac{1}{2}\right)\mathrm{d}\theta$$

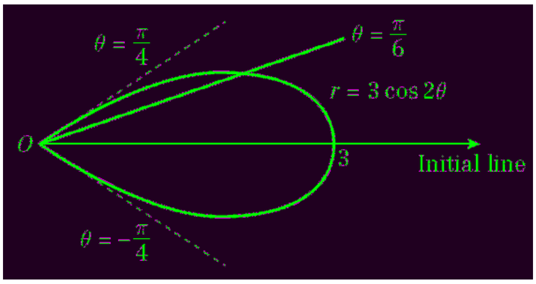
$$=\frac{a^2}{4}\left(\frac{\sin 6\theta}{6}+\theta\right)$$

$\sin\left(6\times\frac{\pi}{6}\right)=\sin\pi=0$

$$A=\frac{a^2}{4}\left[\frac{\sin 6\theta}{6}+\theta\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}=\frac{a^2}{4}\left[\frac{1}{6}(0-0)+\frac{\pi}{6}-\left(-\frac{\pi}{6}\right)\right]$$

$$=\frac{a^2}{4}\times\frac{\pi}{3}=\frac{\pi}{12}a^2$$

46 a



At $\theta=-\frac{\pi}{4}$, $r=0$, As θ increases, r increases until $\theta=0$. For $\theta=0$, $3\cos 2\theta$ has its greatest value of 3. After that, as θ increases, r decreases to 0 at $\theta=\frac{\pi}{4}$.

46 b $A=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} r^2\,\mathrm{d}\theta$

$$\frac{1}{2}\int r^2\,\mathrm{d}\theta=\frac{1}{2}\int 9\cos^2 2\theta\,\mathrm{d}\theta$$

$$=\frac{9}{2}\int\left(\frac{\cos 4\theta}{2}+\frac{1}{2}\right)\mathrm{d}\theta=\frac{9}{4}\int(\cos 4\theta+1)\,\mathrm{d}\theta$$

$$=\frac{9}{4}\left[\frac{\sin 4\theta}{4}+\theta\right]$$

Using $\cos 2A=2\cos^2 A-1$ with $A=2\theta$.

$$A=\frac{9}{4}\left[\frac{\sin 4\theta}{4}+\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$=\frac{9}{4}\left[\frac{1}{4}\left(0-\frac{\sqrt{3}}{2}\right)+\left(\frac{\pi}{4}-\frac{\pi}{6}\right)\right]$$

$$=-\frac{9\sqrt{3}}{32}+\frac{3\pi}{16}=\frac{3}{32}(2\pi-3\sqrt{3})$$

$\sin\left(4\times\frac{\pi}{6}\right)=\sin\frac{2\pi}{3}=\frac{\sqrt{3}}{2}$

c Let $y=r\sin\theta=3\cos 2\theta\sin\theta$

$$\frac{\mathrm{d}y}{\mathrm{d}\theta}=-6\sin 2\theta\sin\theta+3\cos 2\theta\cos\theta=0$$

$$2\sin 2\theta\sin 2\theta=\cos 2\theta\cos\theta$$

$$\frac{\sin 2\theta\sin\theta}{\cos 2\theta\cos\theta}=\tan 2\theta\tan\theta=\frac{1}{2}$$

Where the tangent at a point is parallel to the initial line, the distance y from the point to the initial line has a stationary value. You find the polar coordinate θ of such a point by finding the value of θ for which $y=r\sin\theta$ has a stationary value.

$$\frac{2\tan^2\theta}{1-\tan^2\theta}=\frac{1}{2}$$

Using $\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}$.

$$4\tan^2\theta=1-\tan^2\theta$$

$$5\tan^2\theta=1$$

$$\tan\theta=\frac{1}{\sqrt{5}}$$

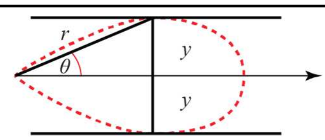
One value of $\tan\theta$ is sufficient to complete the question. r is not needed.

The distance between the two tangents is given by

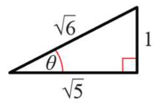
$$2y=2r\sin\theta=6\cos 2\theta\sin\theta=6(2\cos^2\theta-1)\sin\theta$$

$$=6\times\left(2\times\frac{5}{6}-1\right)\times\frac{1}{\sqrt{6}}=6\times\frac{2}{3}\times\frac{1}{\sqrt{6}}$$

$$=\frac{2\sqrt{6}}{3}$$



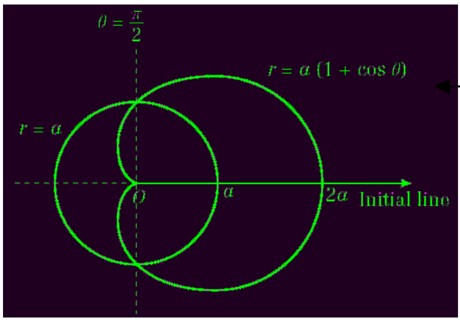
This sketch shows you that the distance between the two tangents parallel to the initial line is given by $2y=2r\sin\theta$.



As $(\sqrt{5})^2+1^2=(\sqrt{6})^2$, if $\tan\theta=\frac{1}{\sqrt{5}}$,

then $\sin\theta=\frac{1}{\sqrt{6}}$ and $\cos\theta=\frac{\sqrt{5}}{\sqrt{6}}$.

47 a



$r = a(1+\cos\theta)$ is a cardioid and $r = a$ is a circle centre O, radius a.

b Let $y = r\sin\theta = a(1+\cos\theta)\sin\theta$

$$= a\sin\theta + a\cos\theta\sin\theta = a\sin\theta + \frac{a}{2}\sin 2\theta$$

$$\frac{dy}{d\theta} = a\cos\theta + a\cos 2\theta = 0$$

$$\cos 2\theta + \cos\theta = 2\cos^2\theta - 1 + \cos\theta = 0$$

$$2\cos^2\theta + \cos\theta - 1 = (2\cos\theta - 1)(\cos\theta + 1) = 0$$

$$\cos\theta = \frac{1}{2},\ \cos\theta = -1$$

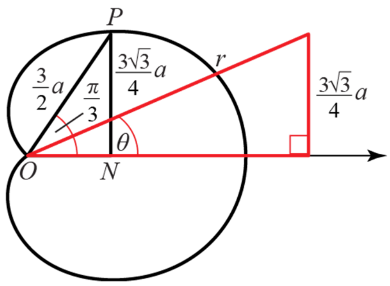
$$\theta = \pm\frac{\pi}{3},\ \theta = \pi$$

At $\theta = \frac{\pi}{3}$,

$$r = a\left(1+\cos\frac{\pi}{3}\right) = a\left(1+\frac{1}{2}\right) = \frac{3}{2}a$$

And $y = r\sin\frac{\pi}{3} = \frac{3}{2}a \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4}a$

Where the tangent at a point is parallel to the initial line, the distance y from the point to the initial line has a stationary value. You find the polar coordinates θ of such points by finding the values of θ for which $y = r\sin\theta$ has stationary values.



You find the distance (labelled PN in the diagram above) from the point where the tangent meets the curve to the initial line.

The polar equation of the tangent is given by

$$r\sin\theta = \frac{3\sqrt{3}}{4}a$$

$$r = \frac{3a\sqrt{3}}{4}\operatorname{cosec}\theta$$

The polar equation is found by trigonometry in the triangle marked in red on the diagram above.

Similarly at $\theta = -\frac{\pi}{3}$, the equation of the

tangent is $r = -\frac{3a\sqrt{3}}{4}\operatorname{cosec}\theta$.

At $\theta = \pi$, the equation of the tangent is

$\theta = \pi$.

It is easy to overlook this case. The half-line $\theta = \pi$ does touch the cardioid at the pole.

47 c The circle and the cardioids meet when

$a = a(1+\cos\theta) \Rightarrow \cos\theta = \theta$

$\theta = \pm\frac{\pi}{2}$

To find the area of the cardioid between

$\theta = -\frac{\pi}{2}$ and $\theta = \frac{\pi}{2}$

$A = 2\times\frac{1}{2}\int_0^{\frac{\pi}{2}} r^2 \, \mathrm{d}\theta$

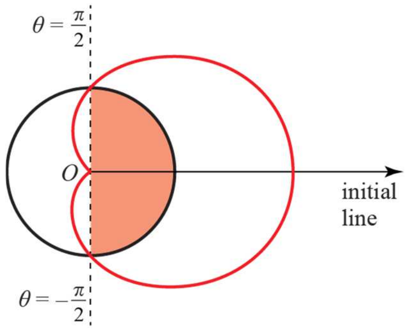
The total area is twice the area above the initial line.

$$\int r^2 \mathrm{d}\theta = \int a^2(1+\cos\theta)^2 \mathrm{d}\theta = \int a^2(1+2\cos\theta+\cos^2\theta)\,\mathrm{d}\theta$$

$$= a^2\int\left(1+2\cos\theta+\frac{1}{2}\cos 2\theta+\frac{1}{2}\right)\mathrm{d}\theta$$

$$= a^2\int\left(\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right)\mathrm{d}\theta$$

$$= a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]$$



$$A = a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= a^2\left(\frac{3\pi}{4}+2\right)$$

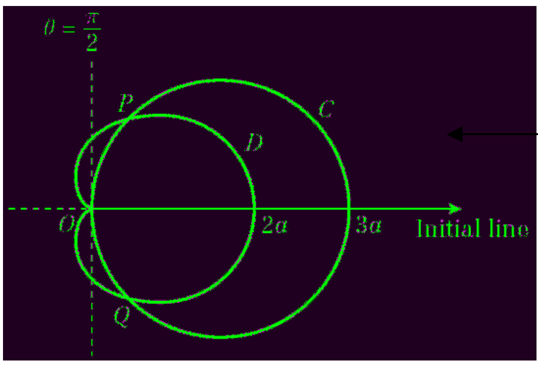
The required area is A less half of the circle

$$\left(\frac{3\pi}{4}+2\right)a^2 - \frac{1}{2}\pi a^2 = \frac{1}{4}\pi a^2 + 2a^2$$

$$= \left(\frac{\pi+8}{4}\right)a^2, \text{ as required.}$$

The area you are asked to find is inside the cardioid and outside the circle. You find it by subtracting the shaded semi-circle from the area of the cardioid bounded by the half-lines $\theta = \frac{\pi}{2}$ and $\theta = -\frac{\pi}{2}$.

48 a



The curve C is a circle of diameter $3a$ and the curve D is a cardioid.
The points of intersection of C and D have been marked on the diagram. The question does not specify which is P and which is Q. They could be interchanged. This would make no substantial difference to the solution of the question.

48 b The points of intersection of C and D are given by

$$3\cancel{a}\cos\theta = \cancel{a}(1+\cos\theta)$$

$$2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2}$$

$$\theta = \pm\frac{\pi}{3}$$

In this question $-\frac{\pi}{2} \leqslant \theta < \frac{\pi}{2}$.

Where $\cos\theta = \frac{1}{2}$

$$r = 3a\cos\frac{\pi}{3} = 3a \times \frac{1}{2} = \frac{3}{2}a$$

$$P: \left(\frac{3}{2}a, \frac{\pi}{3}\right), Q: \left(\frac{3}{2}a, -\frac{\pi}{3}\right)$$

c The area between D, the initial line and OP is given by

$$A_1 = \frac{1}{2}\int_0^{\frac{\pi}{3}} r^2 \mathrm{d}\theta$$

$$\int r^2\,\mathrm{d}\theta = \int a^2(1+\cos\theta)^2\,\mathrm{d}\theta = a^2\int(1+2\cos\theta+\cos^2\theta)\,\mathrm{d}\theta$$

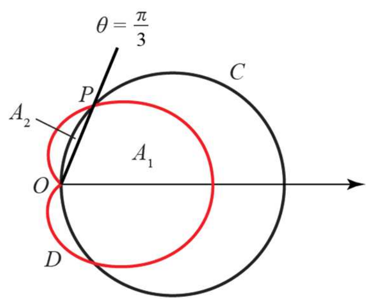
$$= a^2\int\left(1+2\cos\theta+\frac{1}{2}\cos 2\theta+\frac{1}{2}\right)\mathrm{d}\theta$$

$$= a^2\int\left(\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right)\mathrm{d}\theta$$

$$= a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]$$

$$A_1 = \frac{1}{2}\times a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{3}}$$

$$= \frac{a^2}{2}\left[\frac{\pi}{2}+\sqrt{3}+\frac{\sqrt{3}}{8}\right] = \frac{a^2}{16}(4\pi+9\sqrt{3})$$



By the symmetry of the figure, to find the area inside C but outside D, you subtract two areas A_1 and two areas A_2 from the area inside C. C is a circle of radius $\frac{3a}{2}$.

d Let the smaller area enclosed by C and the half-line $\theta = \frac{\pi}{3}$ be A_2.

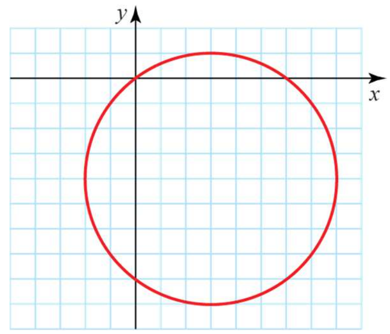
$$R = \pi\left(\frac{3a}{2}\right)^2 - 2A_1 - 2A_2$$

$$= \frac{9a^2\pi}{4} - \frac{2a^2}{16}(4\pi+9\sqrt{3}) - \frac{6a^2}{16}(2\pi-3\sqrt{3})$$

This is twice the area you are given in the question.

$$= \frac{9a^2\pi}{4} - \frac{\pi a^2}{2} - \cancel{\frac{9\sqrt{3}a^2}{8}} - \frac{3\pi a^2}{4} + \cancel{\frac{9\sqrt{3}a^2}{8}} = \pi a^2, \text{ as required.}$$

49 **a** The locus is a circle of centre $3-4\mathrm{i}$ and radius 5, so the Argand diagram is the following:



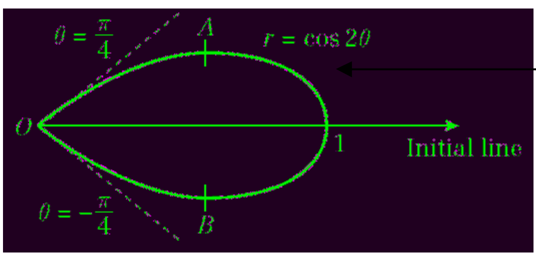
b Suppose z is such that $|z-3+4\mathrm{i}|=5$. Then assuming $z=r(\cos\theta+\mathrm{i}\sin\theta)$ we get that $|r\cos\theta-3+\mathrm{i}(r\sin\theta+4)|=5$; but the magnitude of this complex number is given by $\sqrt{(r\cos\theta-3)^2+(r\sin\theta+4)^2}$, so we get the following:

$$r^2\cos^2\theta-6r\cos\theta+9+r^2\sin^2\theta+8r\sin\theta+16=25$$
$$r^2+25-6r\cos\theta+8r\sin\theta=25$$
$$r=6\cos\theta-8\sin\theta$$

c The area of A is the area of the circle minus the areas that are enclosed in the fourth Cartesian quadrant. Now consider the 63circular sector enclosed between the radii that intersect the circle on the real line. The intersections between the circle and the real line are the origin and 6. Then since the circle has centre $3-4i$, if we interpret this in the Cartesian plane we can find the angle between the radii: it is $\arccos\left(\frac{16-9}{25}\right)$, as the cosine of the angle is given by the ratio between the inner product and the product of the magnitudes of the two radii, seen as vector. Then the area of the circular sector is $\frac{\arccos\left(\frac{7}{25}\right)\cdot 25}{2}$. From this we subtract the area of the triangle formed by the two real points and the circle, which is 12. With the same procedure applied to the complex line we find that the arc between the origin and $-8i$ encloses an area of $\frac{\arccos\left(-\frac{7}{25}\right)\cdot 25}{2}-12$. Then the area of A is:

$$25\pi-\left(\frac{\arccos\left(-\frac{7}{25}\right)\cdot 25}{2}-12\right)-\left(\frac{\arccos\left(\frac{7}{25}\right)\cdot 25}{2}-12\right)$$
$$=78.5-11.2-4.1=63.3$$

50 a



At $\theta=-\frac{\pi}{4}$, $r=0$. As θ increases, r increases until $\theta=0$. For $\theta=0$, $\cos 2\theta$ has its greatest value of 1. After that, as θ increases, r decreases to 0 at $\theta=\frac{\pi}{4}$.

50 b $y = r\sin\theta = \cos 2\theta \sin\theta$

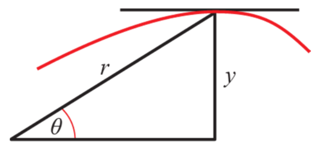
$\frac{dy}{d\theta} = -2\sin 2\theta \sin\theta + \cos 2\theta \cos\theta = 0$

$-4\sin\theta\cos\theta\sin\theta + (1-2\sin^2\theta)\cos\theta = 0$

$\cos\theta(-4\sin^2\theta + 1 - 2\sin^2\theta) = 0$

At A and B, $\cos\theta \neq 0$

$6\sin^2\theta = 1$



Where the tangent at a point is parallel to the initial line, the distance y from the point to the initial line has a stationary value. The diagram above shows that $y = r\sin\theta$. You find the polar coordinates θ of the points by finding the values of θ for which $r\sin\theta$ has a maximum or minimum value.

$\sin\theta = \pm\frac{1}{\sqrt{6}}$

$\theta = \pm 0.420\,534\ldots$

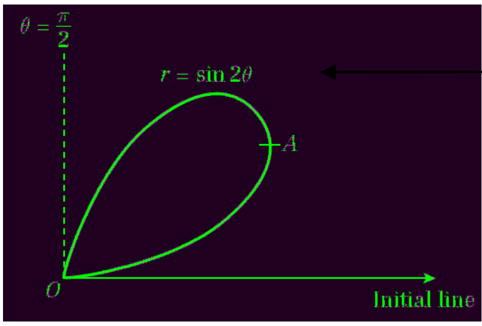
$r = \cos 2\theta = 1 - 2\sin^2\theta = 1 - \frac{2}{6} = \frac{2}{3}$

To 3 significant figures, the polar coordinates of A and B are

(0.667, 0.421) and (0.667, −0.421).

r has an exact value but the question specifically asks for 3 significant figures. Unless the question specifies otherwise, in polar coordinates, you should always give the value of the angle in radians.

51 a



At $\theta = 0$, $r = 0$. As θ increases, r increases until $\theta = \frac{\pi}{4}$. For $\theta = \frac{\pi}{4}$, $\sin 2\theta$ has its greatest value of 1. After that, as θ increases, r decreases to $\sin\left(2\times\frac{\pi}{2}\right) = \sin\pi = 0$ at $\theta = \frac{\pi}{2}$.

51 b $x = r\cos\theta = \sin 2\theta \cos\theta$

$$\frac{dx}{d\theta} = 2\cos 2\theta\cos\theta - \sin 2\theta \sin\theta$$
$$= 2(2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta\sin\theta$$
$$= 2(2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta\cos\theta$$
$$= 4\cos^3\theta - 2\cos\theta - 2(1-\cos^2\theta)\cos\theta$$
$$= 6\cos^3\theta - 4\cos\theta = 0$$
$$2\cos\theta(3\cos^2\theta - 2) = 0$$

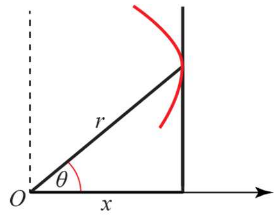
At A, $\cos\theta \neq 0$

$$\cos^2\theta = \frac{2}{3}$$
$$\cos\theta = \left(\frac{2}{3}\right)^{\frac{1}{2}}, \text{ for } 0 \leqslant \theta \leqslant \frac{\pi}{2}$$
$$\theta = 0.615\,479\ldots$$

By calculator

$r = \sin 2\theta = 0.942\,809\ldots$

To 3 significant figures, the coordinates of A are $(0.943, 0.615)$

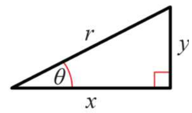


Where the tangent at a point is parallel to $\theta = \frac{\pi}{2}$ (which is the same as being perpendicular to the initial line), the distance x from the half line $\theta = \frac{\pi}{2}$ has a stationary value. The diagram above shows that $x = r\cos\theta$. You find the polar coordinates θ of such points by finding the values of θ for which $r\cos\theta$ has a maximum or minimum value.

52 a $r = 6\cos\theta$

Multiplying the equation by r

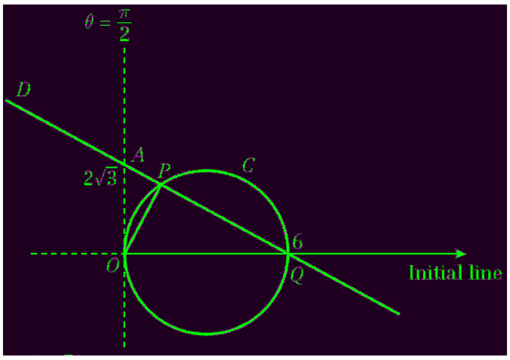
$$r^2 = 6r\cos\theta$$
$$x^2 + y^2 = 6x$$
$$x^2 - 6x + 9 + y^2 = 0$$
$$(x-3)^2 + y^2 = 9$$
$$r = 3\sec\left(\frac{\pi}{3} - \theta\right)$$
$$3 = r\cos\left(\frac{\pi}{3} - \theta\right) = r\cos\frac{\pi}{3}\cos\theta + r\sin\frac{\pi}{3}\sin\theta$$
$$= \frac{1}{2}r\cos\theta + \frac{\sqrt{3}}{2}r\sin\theta$$
$$= \frac{1}{2}x + \frac{\sqrt{3}}{2}y$$
$$x + \sqrt{3}y = 6$$



This diagram illustrates the relations between polar and Cartesian coordinates. The relations you need to solve the question are $r^2 = x^2 + y^2$, $x = r\cos\theta$ and $y = r\sin\theta$.

This is an acceptable answer but putting the equation into a form which shows that the curve is a circle, centre (3, 0) and radius 3, helps you to draw the sketch in part **b**.

52 b



The initial line is the positive x-axis and the half-line $u = \frac{\pi}{2}$ is the positive y-axis.

At $x = 0$, $x + \sqrt{3}y = 6$ gives $y = \frac{6}{\sqrt{3}} = 2\sqrt{3}$.

c By inspection, the polar coordinates of Q are (6, 0)

The question does not say which point is P and which is Q. You can choose which is which.

$\angle OPQ = 90^\circ$

The angle in a semi-circle is a right angle.

In the triangle OAQ

$$\tan AQO = \frac{OA}{OQ} = \frac{2\sqrt{3}}{6} = \frac{\sqrt{3}}{3} \Rightarrow \angle AQO = 30^\circ$$

In the triangle OPQ

$$OP = OQ \sin PQO = 6 \sin 30^\circ = 3$$

$$\angle POQ = 180^\circ - 90^\circ - 30^\circ = 60^\circ = \frac{\pi}{3}$$

Hence the polar coordinates of P are

$$(OP, \angle POQ) = \left(3, \frac{\pi}{3}\right)$$

53 $A = \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2 \,\mathrm{d}\theta$

You need to know the formula for the area of polar curves $A = \frac{1}{2}\int r^2 \,\mathrm{d}\theta$. In this question, the diagram shows that the limits are 0 and $\frac{\pi}{2}$.

$$\frac{1}{2}\int r^2 \,\mathrm{d}\theta = \frac{1}{2}\int a^2 \sin 2\theta \,\mathrm{d}\theta$$

$$= \frac{a^2}{2}\left[-\frac{\cos 2\theta}{2}\right]$$

$$A = \frac{a^2}{4}\left[-\cos 2\theta\right]_0^{\frac{\pi}{2}} = \frac{a^2}{4}[1 - (-1)]$$

$$= \frac{1}{2}a^2$$

$\cos\left(2 \times \frac{\pi}{2}\right) = \cos \pi = -1$ and $\cos 0 = 1$.

54 $A = \frac{1}{2}\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} r^2 \mathrm{d}\theta$

The lower limit, $\frac{\pi}{8}$, is given by the polar equation of m. The upper limit, $\frac{\pi}{4}$, can be identified from the domain of definition, $0 \leqslant \theta \leqslant \frac{\pi}{4}$ given in the question and the diagram.

$$\frac{1}{2}\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} r^2 \mathrm{d}\theta = \frac{1}{2}\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} 16a^2\cos^2 2\theta \mathrm{d}\theta$$

$$= 8a^2\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \cos^2 2\theta \mathrm{d}\theta$$

$$= 8a^2\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \left(\frac{1}{2} + \frac{1}{2}\cos 4\theta\right)\mathrm{d}\theta$$

Using $\cos 2A = 2\cos^2 A - 1$ with $A = 2\theta$.

$$= 4a^2\left[\theta + \frac{\sin 4\theta}{4}\right]_{\frac{\pi}{8}}^{\frac{\pi}{4}}$$

$$A = 4a^2\left[\theta + \frac{\sin 4\theta}{4}\right]_{\frac{\pi}{8}}^{\frac{\pi}{4}}$$

$$= 4a^2\left[\left(\frac{\pi}{4} - \frac{\pi}{8}\right) + \left(0 - \frac{1}{4}\right)\right]$$

$\sin\left(4 \times \frac{\pi}{4}\right) = \sin \pi = 0$ and $\sin\left(4 \times \frac{\pi}{8}\right) = \sin\frac{\pi}{2} = 1$.

$$= 4a^2\left[\frac{\pi}{8} - \frac{1}{4}\right]$$

$$= \tfrac{1}{2}a^2(\pi - 2)$$

55 $A = 2 \times \frac{1}{2}\int_0^{\pi} r^2 \mathrm{d}\theta$

The method used here is to find twice the area above the initial line.

$$= \int_0^{\pi} a^2\left(1 + \frac{1}{2}\cos\theta\right)^2 \mathrm{d}\theta$$

$$= a^2\int_0^{\pi}\left(1 + \cos\theta + \frac{1}{4}\cos^2\theta\right)\mathrm{d}\theta$$

Use $\cos 2\theta = 2\cos^2\theta - 1$.

$$= a^2\int_0^{\pi}\left(1 + \cos\theta + \frac{1}{8}\cos 2\theta + \frac{1}{8}\right)\mathrm{d}\theta$$

$$= a^2\int_0^{\pi}\left(\frac{9}{8} + \cos\theta + \frac{1}{8}\cos 2\theta\right)\mathrm{d}\theta$$

$$= a^2\left[\frac{9}{8}\theta + \sin\theta + \frac{\sin 2\theta}{16}\right]_0^{\pi}$$

As $\sin\pi = \sin 2\pi = 0$ and $\sin 0 = 0$, all of the terms are zero at both the lower and the upper limit except for $\frac{9}{8}\theta$, which has a non-zero value at π.

$$= a^2 \times \frac{9}{8}\pi = \frac{9}{8}\pi a^2$$

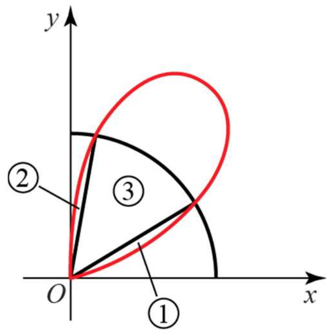
56 a The curves intersect at

$\frac{1}{2} = \sin 2\theta$

$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$\theta = \frac{\pi}{12}, \frac{5\pi}{12}$

b



The shaded area can be broken up into three parts. You can find the small areas labelled (**1**) and (**2**), which are equal in area, by integration. The larger area is a sector of a circle and you find this using $A = \frac{1}{2}r^2\theta$, where θ is in radians.

The area of the sector (**3**) is given by

$$A_3 = \frac{1}{2} \times \left(\frac{1}{2}\right)^2 \times \frac{\pi}{3} = \frac{\pi}{24}$$

The radius of the sector is $\frac{1}{2}$ and the angle is $\frac{5\pi}{12} - \frac{\pi}{12} = \frac{\pi}{3}$.

The area of (**1**) is given by

$$A_1 = \frac{1}{2}\int_0^{\frac{\pi}{12}} r^2 \, d\theta$$

Using $\cos 2A = 1 - 2\sin^2 A$ with $A = 2\theta$.

$$\frac{1}{2}\int \sin^2 2\theta \, d\theta = \frac{1}{2}\int \left(\frac{1}{2} - \frac{1}{2}\cos 4\theta\right) d\theta$$

$$= \frac{1}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]$$

$$A_1 = \frac{1}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{12}}$$

$\sin\left(4 \times \frac{\pi}{12}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$$= \frac{1}{4}\left[\frac{\pi}{12} - 0 - \frac{1}{4}\left(\frac{\sqrt{3}}{2} - 0\right)\right]$$

$$= \frac{1}{4}\left[\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right]$$

The area of the shaded region is given by

$$2 \times A_1 + A_3 = \frac{1}{2}\left[\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right] + \frac{\pi}{24} = \frac{\pi}{12} - \frac{\sqrt{3}}{16}$$

57 a Let $y = r\sin\theta$

Where the tangent at a point is parallel to the initial line, the distance y from the point to the initial line has a stationary value. You find the polar coordinate θ of the point by finding the value θ for which $y = r\sin\theta$ has a stationary value.

$$y = a(3+\sqrt{5}\cos\theta)\sin\theta$$

$$= 3a\sin\theta + \sqrt{5}a\cos\theta\sin\theta = 3a\sin\theta + \frac{\sqrt{5}a}{2}\sin 2\theta$$

$$\frac{dy}{d\theta} = 3a\cos\theta + \sqrt{5}a\cos 2\theta = 0$$

$$3\cos\theta + \sqrt{5}(2\cos^2\theta - 1) = 0$$

$$2\sqrt{5}\cos^2\theta + 3\cos\theta - \sqrt{5} = 0$$

$$\cos\theta = \frac{-3\pm\sqrt{(9+40)}}{4\sqrt{5}}$$

As $|\cos\theta| \leqslant 1$, you reject the value $-\frac{10}{4\sqrt{5}} \approx -1.118$.

$$= \frac{-3+7}{4\sqrt{5}} = \frac{1}{\sqrt{5}}$$

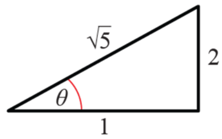
By calculator
$\theta = \pm 1.107$ (3 d.p.)

At $\cos\theta = \frac{1}{\sqrt{5}}$

$$r = a(3+\sqrt{5}\cos\theta) = a\left(3+\sqrt{5}\times\frac{1}{\sqrt{5}}\right) = 4a$$

The polar coordinates are

P: $(4a, 1.107)$, Q: $(4a, -1.107)$



As $1^2 + 2^2 = (\sqrt{5})^2$, the diagram illustrates that if $\cos\theta = \frac{1}{\sqrt{5}}$ then $\sin\theta = \frac{2}{\sqrt{5}}$.

b $PQ = 2y = 2r\sin\theta$

$$= 2\times 4a \times \frac{2}{\sqrt{5}} = \frac{16}{\sqrt{5}}a = 20\text{m, given}$$

$$a = \frac{20\sqrt{5}}{16}\text{m} = \frac{5\sqrt{5}}{4}\text{m}$$

57 c Total area$=2\times\frac{1}{2}\int_{\alpha}^{\beta} r^2\,\mathrm{d}\theta$

$r=a(3+\sqrt{5}\cos\theta)$

$$\text{so area} = a^2\int_0^{\pi}(3+\sqrt{5}\cos\theta)^2\,\mathrm{d}\theta$$

$$= a^2\int_0^{\pi}(9+6\sqrt{5}\cos\theta+5\cos^2\theta)\mathrm{d}\theta$$

$$= a^2\left[9\theta+6\sqrt{5}\sin\theta+\frac{5}{4}\sin 2\theta+\frac{5}{2}\theta\right]_0^{\pi}$$

$$= a^2\left[\frac{23}{2}\theta+6\sqrt{5}\sin\theta+\frac{5}{4}\sin 2\theta\right]_0^{\pi}$$

$$= a^2\left[\frac{23}{2}\pi\right]$$

$$= \frac{23}{2}\pi a^2$$

$$a=\frac{5\sqrt{5}}{4}$$

$$\text{so area} = \frac{23}{2}\left(\frac{5\sqrt{5}}{4}\right)^2\pi$$

$$= \frac{2875}{32}\pi\ \mathrm{m}^2$$

58 a Area $=\frac{1}{2}\int_{-\pi}^{\pi} r^2\,\mathrm{d}\theta$

$$= \frac{1}{2}\int_{-\pi}^{\pi} a^2(1+\cos\theta)^2\,\mathrm{d}\theta$$

$$= \frac{a^2}{2}\int_{-\pi}^{\pi}(1+2\cos\theta+\cos^2\theta)\,\mathrm{d}\theta$$

$$= \frac{a^2}{2}\int_{-\pi}^{\pi}\mathrm{d}\theta + a^2\int_{-\pi}^{\pi}\cos\theta\,\mathrm{d}\theta+\frac{a^2}{2}\int_{-\pi}^{\pi}\cos^2\theta\,\mathrm{d}\theta$$

$$= a^2\pi + a^2\left[\sin\theta\right]_{-\pi}^{\pi}+\frac{a^2}{4}\int_{-\pi}^{\pi}\mathrm{d}\theta+\frac{a^2}{4}\int_{-\pi}^{\pi}\cos 2\theta\,\mathrm{d}\theta$$ using the identity $\cos 2\theta\equiv 2\cos^2\theta-1$

$$= a^2\pi+\frac{a^2\pi}{2}+\frac{a^2}{4}\left[\frac{1}{2}\sin 2\theta\right]_{-\pi}^{\pi}$$

$$= \frac{3\pi a^2}{2}$$

58 b At A and B, $\frac{d}{d\theta}(r\cos\theta) = 0$.

$$\frac{d}{d\theta}(r\cos\theta) = \frac{d}{d\theta}(a\cos\theta(1+\cos\theta))$$
$$= a\frac{d}{d\theta}(\cos\theta + \cos^2\theta)$$
$$= -a\sin\theta + a\frac{d}{d\theta}(\cos^2\theta)$$
$$= -a\sin\theta - 2a\cos\theta\sin\theta, \text{ using the product rule}$$
$$= -a\sin\theta(1+2\cos\theta)$$

Setting this equal to 0 gives:

$\sin\theta(1+2\cos\theta) = 0$

$\sin\theta = 0$ or $\cos\theta = -\frac{1}{2}$

$\therefore\ \theta = 0$ or $\pm\frac{2\pi}{3}$

But $\theta = 0$ is where C intersects the initial line so $\theta = \pm\frac{2\pi}{3}$ at A and B.

$\theta = \pm\frac{2\pi}{3} \Rightarrow r = a\left(1+\cos\frac{2\pi}{3}\right) = \frac{a}{2}$

So the polar coordinates of A and B are A: $\left(\frac{a}{2}, \frac{2\pi}{3}\right)$ and B : $\left(\frac{a}{2}, -\frac{2\pi}{3}\right)$.

c When C intersects the initial line, $r = 2a$.

Therefore, length $WX = 2a$ + length of x–component of vector $\overrightarrow{OA}$

$= 2a + \frac{2}{a}\cos\frac{\pi}{3} = 2a + \frac{a}{4} = \frac{9a}{4}$.

d Area $WXYZ = \frac{9a}{4} \times \frac{3a\sqrt{3}}{2} = \frac{27\sqrt{3}a^2}{8}$.

e Area wasted = Area $WXYZ$ – Area inside C

$$= \frac{27\sqrt{3}a^2}{8} - \frac{3\pi a^2}{2}$$

So when $a = 10$ cm, the area of card wasted is

$\frac{2700\sqrt{3}}{8} - \frac{300\pi}{2} = 113\text{ cm}^2$ (to 3s.f.)

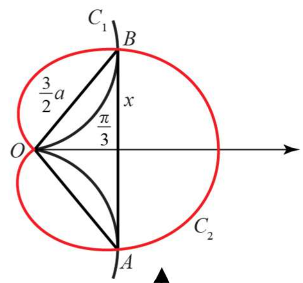
59 a C_1 and C_2 intersect where

$$3\cancel{a}(1-\cos\theta) = \cancel{a}(1+\cos\theta)$$
$$3-3\cos\theta = 1+\cos\theta$$
$$4\cos\theta = 2 \Rightarrow \cos\theta = \frac{1}{2}$$
$$\theta = \pm\frac{\pi}{3}$$

Where $\cos\theta = \frac{1}{2}$

$$r = a(1+\cos\theta) = a\left(1+\frac{1}{2}\right) = \frac{3}{2}a$$

$$A: \left(\frac{3}{2}a, -\frac{\pi}{3}\right),\ B: \left(\frac{3}{2}a, \frac{\pi}{3}\right)$$

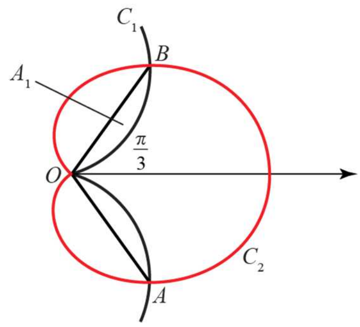


Referring to the diagram,
$\frac{x}{\frac{3}{2}a} = \sin\frac{\pi}{3} \Rightarrow x = \frac{3}{2}a\sin\frac{\pi}{3}$ and
$AB = 2x$.

b $AB = 2\times\frac{3}{2}a\sin\frac{\pi}{3} = 3a\times\frac{\sqrt{3}}{2}$

$= \frac{3\sqrt{3}}{2}a$, as required.

59 c The area A_1 enclosed by OB and C_1 is given by



$$A_1 = \frac{1}{2}\int_0^{\frac{\pi}{3}} r^2 \mathrm{d}\theta$$

$$\int r^2 \mathrm{d}\theta = \int 9a^2(1-\cos\theta)^2 \mathrm{d}\theta = \int 9a^2(1-2\cos\theta+\cos^2\theta)\,\mathrm{d}\theta$$

$$= 9a^2\int\left(1-2\cos\theta+\frac{1}{2}\cos 2\theta+\frac{1}{2}\right)\mathrm{d}\theta$$

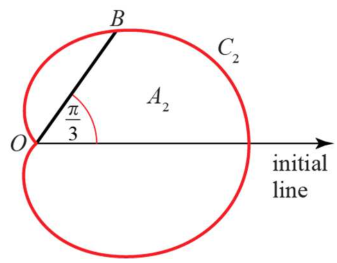
$$= 9a^2\int\left(\frac{3}{2}-2\cos\theta+\frac{1}{2}\cos 2\theta\right)\mathrm{d}\theta$$

$$= 9a^2\left[\frac{3}{2}\theta-2\sin\theta+\frac{1}{4}\sin 2\theta\right]$$

$$A_1 = \frac{1}{2}\times 9a^2\left[\frac{3}{2}\theta-2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{3}}$$

$$= \frac{9}{2}a^2\left[\frac{\pi}{2}-\sqrt{3}+\frac{\sqrt{3}}{8}\right] = \frac{9a^2}{16}(4\pi-7\sqrt{3})$$

The area A_2 enclosed by the initial line, C_2 and OB is given by



$$A_2 = \frac{1}{2}\int_0^{\frac{\pi}{3}} r^2 \mathrm{d}\theta$$

$$\int r^2 \mathrm{d}\theta = \int a^2(1+\cos\theta)^2 \mathrm{d}\theta = a^2\int(1+2\cos\theta+\cos^2\theta)\,\mathrm{d}\theta$$

$$= a^2\int\left(1+2\cos\theta+\frac{1}{2}\cos 2\theta+\frac{1}{2}\right)\mathrm{d}\theta$$

$$= a^2\int\left(\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right)\mathrm{d}\theta$$

$$= a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]$$

$$A_2 = \frac{1}{2}\times a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{3}}$$

$$= \frac{a^2}{2}\left[\frac{\pi}{2}+\sqrt{3}+\frac{\sqrt{3}}{8}\right] = \frac{a^2}{16}(4\pi+9\sqrt{3})$$

The required area R is given by

$$R = 2(A_2 - A_1)$$

$$= 2\left[\frac{a^2}{16}(4\pi+9\sqrt{3})-\frac{9a^2}{16}(4\pi-7\sqrt{3})\right]$$

$$= \frac{2a^2}{16}\left[4\pi+9\sqrt{3}-(36\pi-63\sqrt{3})\right]$$

$$= \frac{a^2}{8}\left[72\sqrt{3}-32\pi\right] = (9\sqrt{3}-4\pi)a^2$$

59 d $\frac{3\sqrt{3}}{2}a = 4.5\,\text{cm}$

$a = \frac{9}{3\sqrt{3}}\,\text{cm} = \sqrt{3}\,\text{cm}$

The area of the badge is

$(9\sqrt{3} - 4\pi)a^2 = (9\sqrt{3} - 4\pi) \times 3\,\text{cm}^2$

$= 9.07\,\text{cm}^2$ (3 s.f.)

You use the result from part **b** to find a and substitute the value of a into the result of part **c**.

60 a $A: (5a,\ 0),\ B: (3a,\ 0)$

For A, at $\theta = 0$, $r = a(3 + 2\cos 0) = a(3 + 2) = 5a$.
For B, at $\theta = 0$, $r = a(5 - 2\cos 0) = a(5 - 2) = 3a$.

b The curves intersect where

$\cancel{a}(3 + 2\cos\theta) = \cancel{a}(5 - 2\cos\theta)$

$4\cos\theta = 2 \Rightarrow \cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

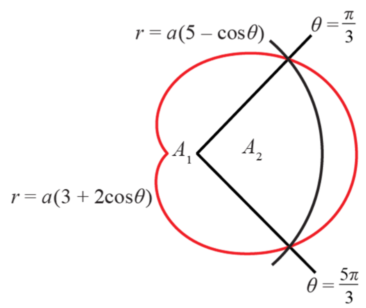
In this question $0 \leq \theta < 2\pi$.

Where $\cos\theta = \frac{1}{2}$

$r = a(3 + 2\cos\theta) = a\left(3 + 2 \times \frac{1}{2}\right) = 4a$

$C: \left(4a, \frac{5\pi}{3}\right),\ D: \left(4a, \frac{\pi}{3}\right)$

60 c The area A_1 enclosed by $r = a(3+2\cos\theta)$ and the half-lines $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$ is given by



The shaded area in the question is the sum of the two areas A_1 and A_2 shown in the diagram above. It is important that you carefully distinguish which curve is which.

$$A_1 = 2\times\frac{1}{2}\int_{\frac{\pi}{3}}^{\pi} r^2\,\mathrm{d}\theta$$

$$\int r^2\,\mathrm{d}\theta = \int a^2(3+2\cos\theta)^2\,\mathrm{d}\theta$$

$$= a^2\int(9+12\cos\theta+4\cos^2\theta)\,\mathrm{d}\theta$$

$$= a^2\int(9+12\cos\theta+2\cos2\theta+2)\,\mathrm{d}\theta$$

$$= a^2\int(11+12\cos\theta+2\cos2\theta)\,\mathrm{d}\theta$$

$$= a^2\left[11\theta+12\sin\theta+\sin2\theta\right]$$

$$A_1 = a^2\left[11\theta+12\sin\theta+\sin2\theta\right]_{\frac{\pi}{3}}^{\pi}$$

$$= a^2\left[11\left(\pi-\frac{\pi}{3}\right)+12\left(0-\frac{\sqrt{3}}{2}\right)+\left(0-\frac{\sqrt{3}}{2}\right)\right]$$

$$= a^2\left[\frac{22\pi}{3}-\frac{13\sqrt{3}}{2}\right]$$

The area A_2 enclosed by $r = a(5-2\cos\theta)$ and the half-lines $\theta = \frac{5\pi}{3}$ and $\theta = \frac{\pi}{3}$ is given by

$$A_2 = 2\times\frac{1}{2}\int_0^{\frac{\pi}{3}} r^2\,\mathrm{d}\theta$$

$$\int r^2\,\mathrm{d}\theta = \int a^2(5-2\cos\theta)^2\,\mathrm{d}\theta = a^2\int(25-20\cos\theta+4\cos^2\theta)\,\mathrm{d}\theta$$

$$= a^2\int(25-20\cos\theta+2\cos2\theta+2)\,\mathrm{d}\theta$$

$$= a^2\int(27-20\cos\theta+2\cos2\theta)\,\mathrm{d}\theta$$

$$= a^2\left[27\theta-20\sin\theta+\sin2\theta\right]$$

The double angle formulae, here $\cos2\theta = 2\cos^2\theta-1$, are used in all questions involving the areas of cardioids.

$$A_2 = a^2\left[27\theta-20\sin\theta+\sin2\theta\right]_0^{\frac{\pi}{3}}$$

$$= a^2\left[27\times\frac{\pi}{3}-20\times\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}\right]$$

$$= a^2\left[\frac{27\pi}{3}-\frac{19\sqrt{3}}{2}\right]$$

The area of the overlapping region is given by

$$A_1 + A_2 = a^2\left(\frac{22\pi}{3}-\frac{13\sqrt{3}}{2}+\frac{27\pi}{3}-\frac{19\sqrt{3}}{2}\right)$$

$$= a^2\left(\frac{49\pi}{3}-16\sqrt{3}\right)$$

$$= \frac{a^2}{3}(49\pi-48\sqrt{3}),\text{ as required.}$$

Challenge

The gradient of the line l is clearly the tangent of the angle $\alpha+\theta$. This gradient can be expressed as follows: since the Cartesian coordinates of P are $(r\cos\theta, r\sin\theta)$, the gradient is $\dfrac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta}$.

Now obviously $\alpha = \alpha+\theta-\theta$, so $\tan\alpha = \tan((\alpha+\theta)-\theta)$. By a formula, this can be solved as follows:

$$\tan((\alpha+\theta)-\theta) = \frac{\tan(\alpha+\theta)-\tan\theta}{1+\tan(\alpha+\theta)\tan\theta} =$$

$$= \frac{\dfrac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta} - \tan\theta}{1+\dfrac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta}\tan\theta} =$$

$$= \frac{\dfrac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta\cos\theta + r\cos^2\theta - \frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta\sin\theta + r\sin^2\theta}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos^2\theta - r\sin\theta\cos\theta}}{1+\dfrac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin^2\theta + r\cos\theta\sin\theta}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos^2\theta - r\sin\theta\cos\theta}} =$$

$$= \frac{r\cos^2\theta + r\sin^2\theta}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos^2\theta + \frac{\mathrm{d}r}{\mathrm{d}\theta}\sin^2\theta} = \frac{r}{\frac{\mathrm{d}r}{\mathrm{d}\theta}}$$